AKT II – Atomic, Nuclear and Particle Physics II

18.3.2021

Standard Model of Particle Physics

Fermions, spin ½, intrinsic parity +1

	Generation I	Generation II	Generation III	weak isospin	electr chrg	color chrg
Quarks	**up** **u** *2.2 MeV*	**charm** **c** *1.3 GeV*	**top** **t** *173 GeV*	+½	+⅔	rgb
	down **d** *4.7 MeV*	**strange** **s** *0.1 GeV*	**bottom** **b** *4.2 GeV*	-½	-⅓	rgb
Leptons	**electron** **e⁻** *0.5 MeV*	**muon** **μ⁻** *0.1 GeV*	**tau** **τ⁻** *1.8 GeV*	-½	-1	
	electron neutrino ν_e *<1.1 eV*	**muon neutrino** ν_μ *<0.2 MeV*	**tau neutrino** ν_τ *<18 MeV*	+½		

(weak isospin: weak interaction; electr chrg: EM interaction; color chrg: strong interaction)

Gauge Bosons, spin 1, intrinsic parity -1

EM	**photon** **γ** *0 GeV*	
strong	**gluon** **g** *0 GeV*	
weak interaction	**W boson** **W±** *80 GeV*	Q=±1
weak interaction	**Z boson** **Z** *91 GeV*	

spin 0: **higgs** **H** *125 GeV*, **parity +1**

Gluons are carrying both color and anti-color. They participate in strong interactions. There are 8 types:
$\frac{r\bar{b}+b\bar{r}}{\sqrt{2}}, \frac{r\bar{g}+g\bar{r}}{\sqrt{2}}, \frac{b\bar{g}+g\bar{b}}{\sqrt{2}}, \frac{r\bar{r}-b\bar{b}}{\sqrt{2}}, -i\frac{r\bar{b}-b\bar{r}}{\sqrt{2}}, -i\frac{r\bar{g}-g\bar{r}}{\sqrt{2}}, -i\frac{b\bar{g}+g\bar{b}}{\sqrt{2}}, \frac{r\bar{r}+b\bar{b}-2g\bar{g}}{\sqrt{2}}$

Lifetime: muon 2 μs, tauon 290 fs

Neutrinos ν_e, ν_μ, and ν_τ are mixtures of 3 fundamental neutrino states with defined masses ν_1, ν_2, and ν_3.

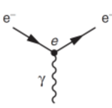
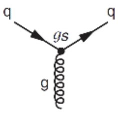
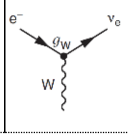
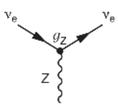
Flavor: u, d, c, s, t, b

Hadrons are bound Quark states

Baryons: Hadrons w. odd number of quarks e.g. p(uud), n(ddu), half-spin

Mesons: Hadrons with even number of quarks (e.g. $q\bar{q}$), integer spin

Interaction Vertices

Electromagnetism	Strong Interaction	Weak Charged Current Interaction	Weak Neutral Interaction
 All charged particles, never changes flavor. $\alpha = 1/137$	 Only Quarks and the gluon itself, never changes flavor. $\alpha_S = 1$	 W± couples charged Leptons with corresp. neutrinos and all Quark combinations so that charge is conserved. Always changes flavor! $\alpha_W = 1/30$	 All Fermions Never changes flavor. $\alpha_Z = 1/30$
Coupling constant g	Determines strength of interaction between gauge boson and fermion = probability of fermion to emit or absorb boson. Scattering process with two vertices: $\mathcal{M} \propto g^2 \Longrightarrow$ Interaction probability $p = \lvert\mathcal{M}\rvert^2 \propto g^4$		
Fine struc. const. α	$\alpha \propto g$; $\alpha_{EM} = \frac{e^2}{4\pi\varepsilon_0\hbar c}$. Intrinsic strength of weak interaction > QED, but because of W-boson's large mass it's smaller.		

Natural Units

Physical Quantity	$[kg, m, s]$	$[\hbar, c, GeV]$	$\hbar = c = 1$	conversion
energy E	$[J] = \left[\frac{kg\, m^2}{s^2}\right]$	$[GeV]$	$[GeV]$	$E[J] = E[eV]\cdot e$
momentum $\vec{p}$	$\left[\frac{kg\, m}{s}\right]$	$\left[\frac{GeV}{c}\right]$	$[GeV]$	$\vec{p}\left[\frac{kg\, m}{s}\right] = \frac{\vec{p}[eV]\cdot e}{c}$
mass m	$[kg]$	$\left[\frac{GeV}{c^2}\right]$	$[GeV]$	$m[kg] = \frac{m[eV]\cdot e}{c^2}$
time t	$[s]$	$\left[\frac{\hbar}{GeV}\right]$	$\left[\frac{1}{GeV}\right]$	$t[s] = t\left[\frac{1}{eV}\right]\frac{\hbar}{e} = t\left[\frac{1}{GeV}\right]\frac{0.2[GeV]10^{-15}[m]}{c[m/s]}$
distance d	$[m]$	$\left[\frac{\hbar c}{GeV}\right]$	$\left[\frac{1}{GeV}\right]$	$d[m] = d\left[\frac{1}{eV}\right]\frac{\hbar c}{e} = d\left[\frac{1}{GeV}\right]0.2[GeV]10^{-15}[m]$
area A	$[m^2]$	$\left[\left(\frac{\hbar c}{GeV}\right)^2\right]$	$\left[\frac{1}{GeV^2}\right]$	$A[m^2] = A\left[\frac{1}{eV^2}\right]\left(\frac{\hbar c}{e}\right)^2 = A\left[\frac{1}{GeV^2}\right](0.2[GeV]10^{-15}[m])^2$

Further Units

Barn $[b]$	$1b = 10^{-28}\,m^2$

$\hbar c$ = 197MeV fm ≈ 0.2GeV fm

$\hbar \approx 10^{-34} Js$, $e \approx 10^{-19} C$, $c \approx 10^8 \frac{m}{s}$

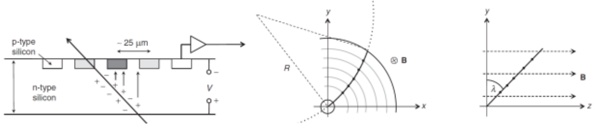
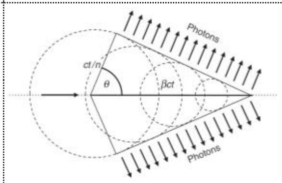
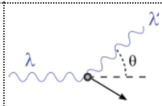
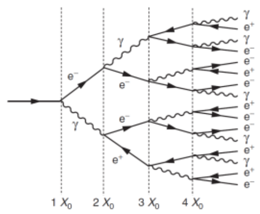
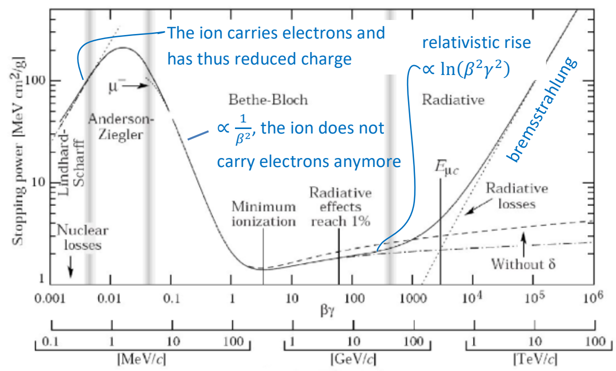
Heavyside-Lorentz: $\hbar = \varepsilon_0 = \mu_0 = 1$

Special Relativity and Four-Vectors

Beta and Gamma	$\gamma = \frac{1}{\sqrt{1-\beta^2}}$; $\beta = \frac{v}{c}$	Lorentz-transformation	Let S′ be the „moving" system, and let S be the „rest" system; i.e. velocity and direction of S′ with respect to S determine magnitude and sign of β. $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$
Active LT Boost in x $S' \to S$:	$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ How does "moving" system S′ look like in "rest" system S? $\boldsymbol{a^\mu = \Lambda^\mu{}_\nu a'^\nu}$	Passive LT Boost in x, $S \to S'$:	$\tilde{\Lambda}^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ How does "rest" system S look like in "moving" S′? $\boldsymbol{a'^\mu = \tilde{\Lambda}^\mu{}_\nu a^\nu}$
4-vector position	$x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \overset{c=1}{=} \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$	Proper Time τ in S′	$ds^2 = ds'^2 \Rightarrow c^2\,dt^2 - dx^2 - dy^2 - dz^2 = c^2\,d\tau^2 \Rightarrow d\tau^2 = \frac{ds^2}{c^2} = \left(1 - \frac{\vec{v}^2}{c^2}\right)dt^2 \Rightarrow$ $d\tau = \frac{1}{\gamma}dt$
4-vector velocity	$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma\begin{pmatrix} c \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \gamma c \\ \gamma\vec{v} \end{pmatrix} \overset{c=1}{=} \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix}$		$u^\mu u_\mu = \gamma^2(c^2 - \vec{v}^2) = c^2 > 0 \Rightarrow$ time-like, invariant
4-vector momentum	$p^\mu = m_0 u^\mu = m_0\begin{pmatrix} \gamma c \\ \gamma\vec{v} \end{pmatrix} = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} m_0 c + \frac{E_{kin}}{c} \\ \vec{p} \end{pmatrix} \overset{c=1}{=} \begin{pmatrix} m_0 + E_{kin} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$		$p^\mu p_\mu = p_0^2 - \vec{p}^2 \equiv \frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2$...invariant $p^\mu p_\mu \overset{c=1}{=} p_0^2 - \vec{p}^2 \equiv E^2 - \vec{p}^2 = m_0^2$ ...invariant
Derivations	$\partial_\mu = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right) \overset{c=1}{=} (\partial_t, \vec{\nabla})$ $\quad\partial^\mu = \begin{pmatrix} \frac{1}{c}\partial_t \\ -\vec{\nabla} \end{pmatrix} \overset{c=1}{=} \begin{pmatrix} \partial_t \\ -\vec{\nabla} \end{pmatrix}$		$\partial_\mu\partial^\mu = \square = \frac{\partial^2}{\partial t^2} - \frac{1}{c^2}\vec{\nabla}^2 \overset{c=1}{=} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$
Energy:	massive particle: $E = \sqrt{m_0^2 c^4 + \vec{p}^2 c^2} \overset{c=1}{=} \sqrt{m_0^2 + \vec{p}^2}$		massless particle: $E = \lvert\vec{p}\rvert c \overset{c=1}{=} \lvert\vec{p}\rvert$

$E = \gamma m_0$
$\vec{p} = \gamma m_0 \vec{v} = \gamma m_0 \vec{\beta} c \overset{c=1}{=} \gamma m_0 \vec{\beta}$
$\vec{v} = \frac{\vec{p}}{\gamma m_0} = \frac{\vec{p}}{E}$

Particle Accelerators and Detectors

$\overset{p\ \ e}{\rightarrow\leftarrow}$ Collider (e.g. HERA):	Center-of-mass frame: $p_p^\mu \overset{c=1}{=} \begin{pmatrix} m_p + E_{kin}^p \\ p_p \end{pmatrix} \approx \begin{pmatrix} E_{kin}^p \\ p_p \end{pmatrix} \approx \begin{pmatrix} 920GeV \\ 920GeV \end{pmatrix}$; $p_e^\mu \overset{c=1}{=} \begin{pmatrix} E_{kin}^e \\ p_e \end{pmatrix} \approx \begin{pmatrix} 27.5GeV \\ -27.5GeV \end{pmatrix}$ $p_{tot}^\mu = p_p^\mu + p_e^\mu = \begin{pmatrix} 947.5GeV \\ 892.5GeV \end{pmatrix}$ Available Energy: $\sqrt{s} = \sqrt{p_{tot}^\mu p_\mu^{tot}} = \sqrt{(947.5^2 - 892.5^2)} = 318\ GeV$
Fixed Target proton: what electron energy is required for same s?	proton rest frame: $p_p^\mu \overset{c=1}{=} \begin{pmatrix} m_p \\ 0 \end{pmatrix}$; electron moves: $p_e^\mu \overset{c=1}{=} \begin{pmatrix} E_{kin}^e \\ p_e \end{pmatrix}$; $p_{tot}^\mu = p_p^\mu + p_e^\mu = \begin{pmatrix} m_p + E_{kin}^e \\ p_e \end{pmatrix}$ Available Energy: $\sqrt{s} = \sqrt{p_{tot}^\mu p_\mu^{tot}} = \sqrt{(m_p + E_{kin}^e)^2 - p_e^2} \approx \sqrt{(m_p + E_{kin}^e)^2 - (E_{kin}^e)^2} = \sqrt{m_p^2 + 2m_p E_{kin}^e + (E_{kin}^e)^2 - (E_{kin}^e)^2}$ $\sqrt{s} = \sqrt{m_p^2 + 2m_p E_{kin}^e} \approx \sqrt{2m_p E_{kin}^e} \Longrightarrow E_{kin}^e = \frac{s}{2m_p} \Longrightarrow$ would require electron energy $E_{kin}^e = \frac{318^2}{2\cdot 1} = 50\,500\ GeV$ for same s
LHC resolution	$E = h\nu \approx h\frac{c}{\lambda} \Longrightarrow \lambda = \frac{hc}{E}$; $E_{LHC} = 14TeV \Longrightarrow \lambda = 10^{-19}m$ (quarks: $10^{-17}m$) — Interactions at ATLAS, CMS, ALICE, LHCb
LINAC	A voltage generator induces EM field inside the RF cavities with 400MHz. LHC: $8 \times 2\ MeV$
Cyclic accel.	2 types of magnets: Dipol magnets for beam "bending", quadrupole magnets for focusing (only in one axis!)
Synchrotron	Bremsstrahlung energy loss $E = \frac{4\pi}{3}\frac{e^2\beta^2\gamma^4}{R}$
Detecting momentum	Measuring momentum of charged particle by detecting deflection through Lorentz force (easy compared to energy detection) $F_Z = F_L \Longrightarrow m\omega^2 r = eBv \vert v = r\omega \Longrightarrow \omega = \frac{v}{r} \Longrightarrow m\frac{v^2}{r^2}r = eBv \Longrightarrow m\frac{v}{r} = eB \Longrightarrow \frac{p}{r} = eB \Longrightarrow \boxed{p = eBr}$
Tracking detector	(1) Particle moves through gaseous substance, liberates electrons, which drift in an electric field towards sense wires. (2) Particle moves through doped silicon waver, and generate electron-hole pairs.  The holes drift in direction of the electric field and are collected by pn-junctions. Sensors are shaped in strips. One particle = 10000 electron-hole-pairs. Detectors are placed in cylindrical surfaces. A homogenous $\vec{B}$ field is applied. $p\left[\frac{GeV}{c}\right]\cos(\lambda) = 0.3B[T]R[m]$
Photons: Čerenkov radiation	 Charged article traverses dielectric medium $v > \frac{c}{n} \Longrightarrow \cos(\vartheta) = \frac{ct/n}{vt} = \frac{ct/n}{\beta ct} = \frac{1}{n\beta}$ Photo effect: Small energies — γ gets absorbed, e^- is emitted. $E_e = E_\gamma - E_{binding}$ $\sigma \propto \frac{1}{E^3}$ Compton effect: large energies —  $\sigma \propto \frac{1}{E}$
EM shower	 A high-energy electron interacts in a medium and radiates bremsstrahlung, which turns into a e^-e^+ pair. Also a primary interaction of a high-energy photon will produce e^-e^+ and create a shower. The pair production process continues to produce a cascade of photons, electrons and positrons. The number of particles double after each radiation length X_0 Energy of a particle after x radiation lengths: $\langle E\rangle = \frac{E}{2^x}$ Shower stops, when $\langle E\rangle \leq E_c \Longrightarrow$ $E_c = \frac{E}{2^{x_{max}}} \Rightarrow 2^{x_{max}} = \frac{E}{E_c} \Rightarrow \ln(2^{x_{max}}) = \ln\left(\frac{E}{E_c}\right) \Rightarrow \ln(2)\,x_{max} = \ln\left(\frac{E}{E_c}\right) \Rightarrow \boxed{x_{max} = \frac{\ln(E/E_c)}{\ln(2)}}$
EM calorimeters	Measures Energy of e^-, e^+, γ with $E > 100MeV$. Alternate layers of high-Z material (e.g. lead) and scintillator material. EM-shower in lead layers. Scintillator detects the created electrons. Energy resolution $\frac{\sigma_E}{E} = \frac{3\%-10\%}{\sqrt{E/GeV}}$.
Hadron calorimeter	Measures Energy of hadronic showers. Large. Again, sandwich structure with thick layers of high-density absorbers (eg steel) and thin layers of active material (eg plastic scintillators). Energy resolution $\frac{\sigma_E}{E} = \frac{50\%}{\sqrt{E/GeV}}$
Scintillators	Cost effective way to detect passage of charged particles when precise spatial info is not required. When passing, they leave some of the scintillator molecules in an excited state. The subsequent decay results in emission of UV photons. By adding fluorescent dye, the molecules of the dye absorb the UV photons and emit blue light, which is detected by photomultipliers.
Bethe-Bloch	 Ionisation energy loss per unit length of relativistic charged particle passing through a medium: $-\frac{dE}{dx} = KZ_e^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\left(\frac{2m_ec^2\beta^2\gamma^2}{I_0^2}\right) - \beta^2 - \delta(\beta\gamma)\right]$ K ... constants, Z_e ... charge number particle, Z ... charge number material, A ... mass number material, $I_0 \approx 10Z\ eV$ ... ionization potential $\delta(\beta\gamma)$ ... Energy Correction - independent of particle mass
High energy e^- detection	$\boxed{\frac{dE}{dx} = \left(\frac{dE}{dx}\right)_{radiation} + \left(\frac{dE}{dx}\right)_{ionisation}}$ $\left(\frac{dE}{dx}\right)_{radiation} = -\frac{E}{x_0}$ $\left(\frac{dE}{dx}\right)_{ionisation}$ Bethe-Bloch formula needs modification, because of small electron mass, and e^-e^- QM effects.

Fermi's Golden Rule

Schrödinger: $i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\hat{H}\,\Psi(x,t)\,\Big|\,\hat{H}=\hat{H}_0+\hat{H}'(x,t)\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\left(\hat{H}_0+\hat{H}'(x,t)\right)\Psi(x,t)$ … (1)

$$\Psi(x,t)=\sum_k c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\overset{(1)}{\Rightarrow}i\hbar\frac{\partial}{\partial t}\sum_k c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}=\left(\hat{H}_0+\hat{H}'(x,t)\right)\sum_k c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\Rightarrow$$

$$i\hbar\sum_k\frac{\partial}{\partial t}\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)=\sum_k\hat{H}_0\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)+\sum_k\hat{H}'(x,t)\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)$$

$$i\hbar\sum_k\left(\frac{\partial}{\partial t}c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}-i\frac{E_k}{\hbar}c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)=\sum_k c_k(t)\,\hat{H}_0\,\phi_k(x)\,e^{-iE_kt/\hbar}+\sum_k\hat{H}'(x,t)\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)\Big|\,\hat{H}_0\phi_k=E_k\phi_k$$

$$i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}\phi_k(x)\,e^{-iE_kt/\hbar}+\cancel{\sum_k c_k(t)\,E_k\,\phi_k(x)\,e^{-iE_kt/\hbar}}=\cancel{\sum_k c_k(t)\,E_k\,\phi_k(x)\,e^{-iE_kt/\hbar}}+\sum_k\hat{H}'(x,t)\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)$$

$$i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}\phi_k(x)\,e^{-iE_kt/\hbar}=\sum_k\hat{H}'(x,t)\left(c_k(t)\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)\Big|\text{ let }\Psi(x,0)\overset{!}{=}\phi_i(x)\Rightarrow c_k(0)=\delta_{ik};\text{ for small perturbations also }c_k(t)\approx\delta_{ik}$$

$$i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}\phi_k(x)\,e^{-iE_kt/\hbar}=\sum_k\hat{H}'(x,t)\left(\delta_{ik}\,\phi_k(x)\,e^{-iE_kt/\hbar}\right)=\hat{H}'(x,t)\,\phi_i(x)\,e^{-iE_it/\hbar}\Big|\,\phi_k(x)=|k\rangle,\phi_i(x)=|i\rangle$$

$$i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}|k\rangle e^{-iE_kt}=\hat{H}'(x,t)\,|i\rangle e^{-iE_it/\hbar}\Big|\,\langle f|\cdot\Rightarrow i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}\langle f|k\rangle e^{-iE_kt/\hbar}=\langle f|\,\hat{H}'(x,t)\,|i\rangle e^{-iE_it/\hbar}\Big|\,\langle f|k\rangle=\delta_{fk}$$

$$i\hbar\sum_k\frac{\partial c_k(t)}{\partial t}\delta_{fk}e^{-iE_kt/\hbar}=\langle f|\,\hat{H}'(x,t)\,|i\rangle e^{-iE_it/\hbar}\Rightarrow i\hbar\frac{\partial c_f(t)}{\partial t}e^{-iE_ft/\hbar}=\langle f|\,\hat{H}'(x,t)\,|i\rangle e^{-iE_it/\hbar}\Rightarrow$$

$$\frac{\partial c_f(t)}{\partial t}=\frac{1}{i\hbar}\langle f|\,\hat{H}'(x,t)\,|i\rangle e^{iE_ft/\hbar}e^{-iE_it/\hbar}=\frac{1}{i\hbar}\langle f|\,\hat{H}'(x,t)\,|i\rangle e^{i(E_f-E_i)t/\hbar}\Rightarrow$$

$dc_f(t)=\frac{1}{i\hbar}T_{if}e^{i(E_f-E_i)t/\hbar}\,dt$ with Transition Matrix Element $\boxed{T_{if}=\langle f|\hat{H}'|i\rangle}\Rightarrow$

$c_f(t)=\frac{1}{i\hbar}\int_0^t T_{if}e^{i(E_f-E_i)\tau/\hbar}\,d\tau\,\Big|$ assumption: $T_{if}=T_{if}(x)\Rightarrow c_f(t)=\frac{1}{i\hbar}T_{if}\int_0^t e^{i(E_f-E_i)\tau/\hbar}\,d\tau$ … (2)

$|\Psi(t)\rangle=\sum_k c_k(t)\,e^{-iE_kt/\hbar}|k\rangle\,\Big|\,\langle f|\cdot\Rightarrow\langle f|\,\Psi(t)\rangle=\sum_k c_k(t)\,e^{-iE_kt/\hbar}\langle f|k\rangle=\sum_k c_k(t)\,e^{-iE_kt/\hbar}\delta_{fk}\Rightarrow\langle f|\,\Psi(t)\rangle=c_f(t)\,e^{-iE_ft/\hbar}$ … (3)

Probability for a transition to state $|f\rangle$ after duration T: $p_{fi}=|\langle f|\,\Psi(T)\rangle|^2\overset{(3)}{=}\left|c_f(T)\,e^{-iE_ft/\hbar}\right|^2=c_f^*(T)\,c_f(T)\overset{(2)}{\Rightarrow}$

$p_{fi}(T)=\left(\frac{1}{-i\hbar}T_{if}^*\int_0^T e^{-\frac{i(E_f-E_i)t'}{\hbar}}dt'\right)\left(\frac{1}{i\hbar}T_{if}\int_0^T e^{\frac{i(E_f-E_i)t}{\hbar}}dt\right)\Big|\,\omega_{fi}\overset{\text{def}}{=}\frac{E_f-E_i}{\hbar}$ … (4) $\Rightarrow p_{fi}=\left(\frac{1}{-i\hbar}T_{if}^*\int_0^T e^{-i\omega_{fi}t'}\,dt'\right)\left(\frac{1}{i\hbar}T_{if}\int_0^T e^{i\omega_{fi}t}\,dt\right)$

$p_{fi}(T)=\frac{1}{\hbar^2}\left|T_{if}\right|^2\int_0^T e^{-i\omega_{fi}t'}\,dt'\int_0^T e^{i\omega_{fi}t}\,dt\,\Big|\,\tilde{t}\overset{\text{def}}{=}t-\frac{T}{2}\Rightarrow dt=d\tilde{t};\ t=\tilde{t}+\frac{T}{2}$, same for $t'\to\tilde{t}'$

$p_{fi}(T)=\frac{1}{\hbar^2}\left|T_{if}\right|^2\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{-i\omega_{fi}\left(\tilde{t}'+\frac{T}{2}\right)}d\tilde{t}'\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i\omega_{fi}\left(\tilde{t}+\frac{T}{2}\right)}d\tilde{t}=\frac{1}{\hbar^2}\left|T_{if}\right|^2\cancel{e^{-i\omega_{fi}\frac{T}{2}}e^{i\omega_{fi}\frac{T}{2}}}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{-i\omega_{fi}\tilde{t}'}\,d\tilde{t}'\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i\omega_{fi}\tilde{t}}\,d\tilde{t}$ … (5)

$p_{fi}(T)=\frac{1}{\hbar^2}\left|T_{if}\right|^2 2\frac{\sin(\omega_{fi}T/2)}{\omega_{fi}}2\frac{\sin(\omega_{fi}T/2)}{\omega_{fi}}=\frac{4}{\hbar^2}\left|T_{if}\right|^2\frac{\sin^2(\omega_{fi}T/2)}{\omega_{fi}^2}=\frac{1}{\hbar^2}\left|T_{if}\right|^2\frac{\sin^2(\omega_{fi}T/2)}{(\omega_{fi}/2)^2}$ … (6)

Transition Rate (probability of transition per unit time) for $T\to\infty$: $d\Gamma_{fi}=\lim_{T\to\infty}\frac{1}{T}p_{fi}(T)\overset{(6)}{\Rightarrow}$

$d\Gamma_{fi}=\frac{1}{\hbar^2}\left|T_{if}\right|^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i\omega_{fi}\tilde{t}}\underbrace{\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{-i\omega_{fi}\tilde{t}'}\,d\tilde{t}'}_{2\pi\,\delta(\omega_{fi})}d\tilde{t}=\frac{2\pi}{\hbar^2}\left|T_{if}\right|^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i\omega_{fi}\tilde{t}}\,\delta(\omega_{fi})\,d\tilde{t}$ … (7)

If there are dn accessible states in the energy range $[E_f,E_f+dE_f]$, then the total **not Lorentz invariant** transition rate is

$\Gamma_{fi}=\int d\Gamma_{fi}\,dn=\int d\Gamma_{fi}\frac{dn}{dE_f}dE_f\overset{(7)}{=}\frac{2\pi}{\hbar^2}\left|T_{if}\right|^2\int\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i\omega_{fi}\tilde{t}}\,\delta(\omega_{fi})\,d\tilde{t}\,\frac{dn}{dE_f}dE_f\,\Big|\,\omega_{fi}\overset{\text{def}}{=}\frac{E_f-E_i}{\hbar}$

$\Gamma_{fi}=\frac{2\pi}{\hbar^2}\left|T_{if}\right|^2\int\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i(E_f-E_i)\tilde{t}/\hbar}\frac{dn}{dE_f}\delta\left(\frac{E_f-E_i}{\hbar}\right)d\tilde{t}\,dE_f\,\Big|\,\delta\left(\frac{E_f-E_i}{\hbar}\right)=\hbar\,\delta(E_f-E_i)\Rightarrow$

$\Gamma_{fi}=\frac{2\pi}{\hbar}\left|T_{if}\right|^2\int\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^{i(E_f-E_i)\tilde{t}/\hbar}\frac{dn}{dE_f}\delta(E_f-E_i)\,d\tilde{t}\,dE_f=\frac{2\pi}{\hbar}\left|T_{if}\right|^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}e^0\frac{dn}{dE_f}d\tilde{t}=\frac{2\pi}{\hbar}\left|T_{if}\right|^2\frac{dn}{dE_f}\Big|_{E_i}\underbrace{\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}d\tilde{t}}_{1}$

$\Gamma_{fi}=\frac{2\pi}{\hbar}\left|T_{fi}\right|^2\frac{dn}{dE_f}\Big|_{E_i}\Rightarrow\boxed{\Gamma_{fi}=\frac{2\pi}{\hbar}\left|T_{fi}\right|^2\rho(E_i)\ \text{ with }\ \rho(E_i)=\frac{dn}{dE_f}\Big|_{E_i}}$ … Fermi's Golden Rule (to the first order)

Non-relativistic Phase Space

Final-State particles are represented by plane waves $\Psi(\vec{x},t)=Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}=Ae^{i\left(\frac{\vec{p}}{\hbar}\cdot\vec{x}-\frac{E}{\hbar}t\right)}\overset{\hbar=1}{=}Ae^{i(\vec{p}\cdot\vec{x}-Et)}$ … (9)

Normalization to one particle per one cubic volume of side $a\Rightarrow\int_0^a\int_0^a\int_0^a\Psi^*\Psi\,dx\,dy\,dz=1\overset{(9)}{\Rightarrow}$

$\int_0^a\int_0^a\int_0^a A^*e^{-i(\vec{p}\cdot\vec{x}-Et)}Ae^{i(\vec{p}\cdot\vec{x}-Et)}\,dx\,dy\,dz=1\Rightarrow|A|^2a^3=|A|^2V=1\Rightarrow|A|=\frac{1}{\sqrt{V}}$ … (10) is not Lorentz-invariant!

Boundary condition: $\Psi(x,y,z,t)=\Psi(x+a,y,z,t)$ etc. $\Rightarrow e^{ip_xx}=e^{ip_x(x+a)}\Rightarrow p_xx=p_x(x+a)\pm2\pi n_x\Rightarrow$

$p_xx=p_xx+p_xa\pm2\pi n_x\Rightarrow p_xa=2\pi n_x\Rightarrow p_x=\frac{2\pi n_x}{a},p_y=\frac{2\pi n_y}{a},p_z=\frac{2\pi n_z}{a}$ … (11) $\Rightarrow$

Volume of a single state in p-space: $dp_x=\frac{p_x}{n_x}=\frac{2\pi}{a},dp_y=\frac{p_y}{n_y}=\frac{2\pi}{a},dp_z=\frac{p_z}{n_z}=\frac{2\pi}{a}\Rightarrow\boxed{d^3p=\left(\frac{2\pi}{a}\right)^3=\frac{(2\pi)^3}{V}}$ … (12)

Number of states n in a sphere of radius p in p-space: $n=\frac{\text{volume of sphere in }p\text{-space}}{\text{volume of a single state in }p\text{-space}}=\frac{4\pi p^3}{3}\frac{1}{d^3p}\overset{(12)}{\Rightarrow}$

$n=\frac{4\pi p^3}{3}\frac{V}{(2\pi)^3}\Rightarrow\frac{dn}{dp}=\frac{4\pi p^2}{(2\pi)^3}V$ … (13) Dens. of states: $\rho(E_i)=\frac{dn}{dE}\Big|_{E_i}=\frac{dn}{dp}\left|\frac{dp}{dE}\right|_{E_i}\overset{(13)}{=}\frac{4\pi p^2}{(2\pi)^3}V\left|\frac{dp}{dE}\right|_{E_i}$ … (14)

$E^2=p^2+m^2\Rightarrow p^2=E^2-m^2\Rightarrow p=(E^2-m^2)^{\frac{1}{2}}\Rightarrow\frac{dp}{dE}=\frac{1}{2}(E^2-m^2)^{-\frac{1}{2}}2E=\frac{E}{\sqrt{E^2-m^2}}=\frac{E}{p}=\frac{\gamma m}{\gamma m\beta}\Rightarrow\frac{dp}{dE}=\frac{1}{\beta}$ … (15)

Infinitesimal number of states dn_i for the i-th particle in an cuboid with dimensions $dp_x^i\times dp_y^i\times dp_z^i$ in p-space:

$dn_i=\frac{\text{volume of cuboid}}{\text{volume of a single state in }p\text{-space}}=\frac{dp_x^idp_y^idp_z^i}{d^3p}\overset{(4)}{\Rightarrow}dn=\frac{dp_x^idp_y^idp_z^i}{(2\pi)^3/V}\,\Big|\,V\overset{!}{=}1\Rightarrow dn=\frac{dp_x^idp_y^idp_z^i}{(2\pi)^3}=\frac{d^3\vec{p}_i}{(2\pi)^3}$ … (16)

General non-relativistic expression for N-body phase space (with N-1 independent momenta bc. of momentum conservation):

$dn=dn_1dn_2\dots dn_{N-1}\overset{(16)}{=}\frac{d^3\vec{p}_1}{(2\pi)^3}\cdots\frac{d^3\vec{p}_{N-1}}{(2\pi)^3}=\frac{d^3\vec{p}_1}{(2\pi)^3}\cdots\frac{d^3\vec{p}_{N-1}}{(2\pi)^3}\delta^3(\vec{p}_a-\vec{p}_1-\cdots-\vec{p}_N)\,d^3\vec{p}_N\Rightarrow$

$\boxed{dn=(2\pi)^3\frac{d^3\vec{p}_1}{(2\pi)^3}\cdots\frac{d^3\vec{p}_N}{(2\pi)^3}\delta^3(\vec{p}_a-\vec{p}_1-\cdots-\vec{p}_N)}$ … (17)

Lorentz-invariant matrix element

Non-relativistic: $\int_V\Psi^*\Psi\,d^3x=1$, Lorentz-invariant: $\boxed{\int_V\Psi'^*\Psi'\,d^3x=2E}$ … (18) $\Rightarrow\Psi'=\sqrt{2E}\Psi\Rightarrow$

$\boxed{\mathcal{M}_{fi}=\langle\Psi_1'\Psi_2'\dots|\hat{H}'|\Psi_a'\Psi_b'\dots\rangle=T_{fi}\sqrt{2E_12E_2\dots2E_a2E_b\dots}}$ … (19a) $\boxed{T_{fi}=\mathcal{M}_{fi}\frac{1}{\sqrt{2E_12E_2\dots2E_a2E_b\dots}}}$ … (19b)

Lorentz-Invariant Transition Rate for Two Body Decay

Decay $a \to 1+2$

$(8) \Longrightarrow \Gamma_{fi} = \frac{2\pi}{\hbar}\left|T_{fi}\right|^2 \int_0^\infty \delta(E_f - E_i)\, dn \overset{\hbar=1}{\Longrightarrow} \Gamma_{fi} = 2\pi\left|T_{if}\right|^2 \int_0^\infty \delta(E_f - E_i)\, dn \,\Big|\, E_i = E_a, E_f = E_1 + E_2 \Longrightarrow$

$\Gamma_{fi} = 2\pi\left|T_{fi}\right|^2 \int \delta(E_1 + E_2 - E_a)\, dn = 2\pi\left|T_{if}\right|^2 \int \delta(E_a - E_1 - E_2)\, dn \overset{(17)}{\Longrightarrow}$

$\Gamma_{fi} = (2\pi)^4\left|T_{fi}\right|^2 \int \delta(E_a - E_1 - E_2)\, \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3} \overset{(19b)}{\Longrightarrow}$

$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a - E_1 - E_2)\, \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}}$ … (20) Lorentz invariant transition rate

$dLIPS$: Lorentz invar. phase space and transition rate with 4-vectors

$\boxed{dLIPS = \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\vec{p}_N}{(2\pi)^3 2E_N}}$ … (21) $\overset{(20)}{\Longrightarrow} d\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a - E_1 - E_2)\, \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\, dLIPS$

$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\left|\mathcal{M}_{fi}\right|^2 \int \delta^4(p_a^\mu - p_1^\mu - p_2^\mu)\, dLIPS$ … (22)

$\int \delta(E_i^2 - \vec{p}_i^2 - m_i^2)\, dE_i = \int \delta(\mathrm{f}(E_i))\, dE_i = \int \left|\frac{1}{\mathrm{f}'(E_{root})}\right| \delta(E_i - E_{root})\, dE_i$ with $\mathrm{f}(E_i) = E_i^2 - \vec{p}_i^2 - m^2$ and $\mathrm{f}(E_{root}) = 0$ … (23)

$\mathrm{f}(E_{root}) = 0 \overset{(17)}{\Longrightarrow} E_{root}^2 - \vec{p}_i^2 - m_i^2 = 0 \Longrightarrow E_{root}^2 = \vec{p}_i^2 + m_i^2 \Longrightarrow E_{root} = \sqrt{\vec{p}_i^2 + m_i^2}$ … (24)

$\mathrm{f}'(E_{root}) \overset{(23)}{=} \frac{d}{dE_i}(E_i^2 - \vec{p}_i^2 - m_i^2)\Big|_{E_i = E_{root}} = 2E_i|_{E_i = E_{root}} = 2E_{root} \overset{(24)}{\Longrightarrow} \mathrm{f}'(E_{root}) = 2\sqrt{\vec{p}_i^2 + m_i^2}$ $\Big\} \overset{(23)}{\Longrightarrow}$

$\int \delta(E_i^2 - \vec{p}_i^2 - m_i^2)\, dE_i = \int \frac{1}{2\sqrt{\vec{p}_i^2 + m^2}}\delta\left(E_i - \sqrt{\vec{p}_i^2 + m_i^2}\right) dE_i = \frac{1}{2\sqrt{\vec{p}_i^2 + m^2}} \,\Big|\, \sqrt{\vec{p}_i^2 + m_i^2} = E_i \Longrightarrow \boxed{\int \delta(E_i^2 - \vec{p}_i^2 - m_i^2)\, dE_i = \frac{1}{2E_i}}$ … (25) $\overset{(21)}{\Longrightarrow}$

$dLIPS = \frac{1}{(2\pi)^{3N}}\delta(E_1^2 - \vec{p}_1^2 - m_1^2)\ldots\delta(E_N^2 - \vec{p}_N^2 - m_N^2)\, dE_1 d^3\vec{p}_1 \ldots dE_N d^3\vec{p}_N \,\Big|\, p_i^\mu p_\mu^i = E_i^2 - \vec{p}_i^2$

$\boxed{dLIPS = \frac{1}{(2\pi)^{3N}}\delta(p_1^\mu p_\mu^1 - m_1^2)\ldots\delta(p_N^\nu p_\nu^N - m_N^2)\, d^4p_1 \ldots d^4p_N} \overset{(22)}{\Longrightarrow}$

$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\frac{1}{(2\pi)^6}\left|\mathcal{M}_{fi}\right|^2 \int \delta^4(p_a^\mu - p_1^\mu - p_2^\mu)\, \delta(p_1^\mu p_\mu^1 - m_1^2)\, \delta(p_2^\nu p_\nu^2 - m_2^2)\, d^4p_1 d^4p_2}$ … (26) LI transition rate with 4-vectors

$a \to 1+2$ in center of mass frame

$E_a = m_a, \vec{p}_a = 0 \overset{(20)}{\Longrightarrow} \Gamma_{fi} = \frac{(2\pi)^4}{2m_a}\left|\mathcal{M}_{fi}\right|^2 \int \delta(m_a - E_1 - E_2)\, \delta^3(-\vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \Longrightarrow$

$\boxed{\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2}\int\left|\mathcal{M}_{fi}\right|^2 d\Omega \text{ with } p^* = \frac{1}{2m_a}\sqrt{(m_a^2 - (m_1 + m_2)^2)(m_a^2 - (m_1 - m_2)^2)}}$ … (27)

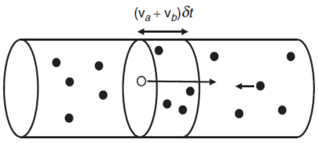
Interaction Rate and Interaction Cross-Section

Cross-section

$\sigma = \frac{\#\,interactions}{\#target\_particles \times time} \cdot \frac{1}{incident\_flux} = \frac{\#\,interactions}{\#target\_particles \times time} \cdot \frac{time \cdot area}{\#incident\_particles}$

diff cr. sect.: $\frac{d\sigma}{d\Omega}$; $d\Omega = \sin(\vartheta)\, d\vartheta\, d\varphi$ — doub diff $\frac{d^2\sigma}{d\Omega dE}$

Interaction probability and rate



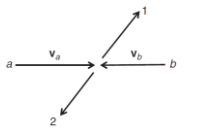
Interaction probability: $dP = \frac{dN_b}{A}\sigma = \frac{n_b\, dV}{A}\sigma = \frac{n_b(v_a + v_b)\, dt\, A}{A}\sigma = n_b(v_a + v_b)\sigma dt$ … (1)

Interaction rate per particle of type a: $r_a = \frac{dP}{dt} \overset{(1)}{=} n_b(v_a + v_b)\sigma = n_b v\sigma$ … (2)

Total interact. rate: rate $= r_a N_a = r_a n_a V \overset{(2)}{=} n_b v\sigma n_a V = (n_a v)(n_b V)\sigma = \phi_a N_b \sigma$ … (3)

with flux of particles type a: $\phi_a = n_a v = n_a(v_a + v_b)$ … (4)

Lorenz invariant flux



$\Gamma_{fi} = \text{rate} \overset{(3)}{=} \phi_a N_b \sigma = \phi_a n_b V \sigma \overset{(4)}{=} n_a(v_a + v_b) n_b V \sigma$ … (5)

normalizing wavefunctions to 1 particle per Volume $\Longrightarrow n_a = n_b = 1 \overset{(5)}{\Longrightarrow} \Gamma_{fi} = (v_a + v_b)V\sigma$ … (6)

normalizing volume to $1 \Longrightarrow \Gamma_{fi} = (v_a + v_b)\sigma \Longrightarrow \boxed{\sigma = \frac{\Gamma_{fi}}{v_a + v_b}}$ … (7)

$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a 2E_b}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \Longrightarrow$

$\Gamma_{fi} = \frac{1}{4E_a E_b}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} \overset{(7)}{\Longrightarrow}$

$\sigma = \frac{1}{4E_a E_b(v_a + v_b)}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$ … (8)

Lorentz invariant flux factor: $\boxed{F = 4E_a E_b(|\vec{v}_a| + |\vec{v}_b|) = 4E_a E_b(v_a + v_b) = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}$ … (9) $\overset{(8)}{\Longrightarrow}$

$\sigma = \frac{1}{F}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$ … (10)

Scattering in center-of-mass frame

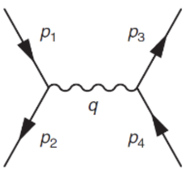
$F = 4E_a^* E_b^*(v_a^* + v_b^*) = 4E_a^* E_b^*\left(\frac{p_a^*}{E_a^*} + \frac{p_b^*}{E_b^*}\right) \Big|\, \vec{p}_a^* = -\vec{p}_b^* \Longrightarrow |\vec{p}_a^*| = |\vec{p}_b^*| = p_i^* \Longrightarrow F = 4E_a^* E_b^*\left(\frac{p_i^*}{E_a^*} + \frac{p_i^*}{E_b^*}\right) = 4E_b^* p_i^* + 4E_a^* p_i^* \Longrightarrow$

$F = 4p_i^*(E_a^* + E_b^*) \overset{(10)}{\Longrightarrow} \Gamma_{fi} = \frac{1}{4p_i^*(E_a^* + E_b^*)}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a^* + E_b^* - E_1^* - E_2^*)\, \delta^3(\vec{p}_a^* + \vec{p}_b^* - \vec{p}_1^* - \vec{p}_2^*)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} \,\Big|\, \vec{p}_a^* = -\vec{p}_b^*$

$\sigma = \frac{1}{4p_i^*(E_a^* + E_b^*)}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(E_a^* + E_b^* - E_1^* - E_2^*)\, \delta^3(-\vec{p}_1^* - \vec{p}_2^*)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} \,\Big|\, E_a^* + E_b^* \overset{\text{def}}{=} \sqrt{s}$

$\sigma = \frac{1}{4p_i^*\sqrt{s}}\frac{1}{(2\pi)^2}\left|\mathcal{M}_{fi}\right|^2 \int \delta(\sqrt{s} - E_1^* - E_2^*)\, \delta^3(\vec{p}_1^* + \vec{p}_2^*)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} = \frac{1}{16\pi^2 p_i^*\sqrt{s}}\frac{p_f^*}{4\sqrt{s}}\int\left|\mathcal{M}_{fi}\right|^2 d\Omega^* \Longrightarrow \boxed{\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\int\left|\mathcal{M}_{fi}\right|^2 d\Omega^*}$ … (11)

Mandelstam Variables



s-channel annihilation

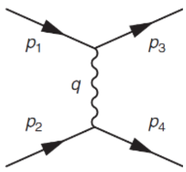
$s = (p_1^\mu + p_2^\mu)^2$

$s = (p_3^\mu + p_4^\mu)^2$

$q = \sqrt{s}$ c.o.m energy

Lorentz-invariant$\Longrightarrow$

$s = (E_1^* + E_2^*)^2$



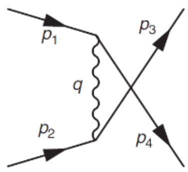
t-channel scattering

$t = (p_1^\mu - p_3^\mu)^2$

$t = (p_2^\mu - p_4^\mu)^2$

$q = \sqrt{t}$

Lorentz-invariant



u-channel scattering

$u = (p_1^\mu - p_4^\mu)^2$

$u = (p_2^\mu - p_3^\mu)^2$

$q = \sqrt{u}$

Lorentz-invariant

$s + u + t = m_1^2 + m_2^2 + m_3^2 + m_4^2$

Klein-Gordon Equation

Derivation of equation	$E^2 \overset{c=1}{=} p^2+m^2 \Rightarrow E^2-p^2=m^2 \Rightarrow \hat E^2-\hat p^2=m^2 \vert \cdot\Psi \Rightarrow (\hat E^2-\hat p^2)\Psi=m^2\Psi \vert \hat E = i\frac{\partial}{\partial t} \Rightarrow \hat E^2 = -\frac{\partial^2}{\partial t^2}\Rightarrow$ $\left(-\frac{\partial^2}{\partial t^2}-\hat p^2\right)\Psi=m^2\Psi \vert \hat p=-i\vec\nabla \Rightarrow \hat p^2=-\vec\nabla^2 \Rightarrow \left(-\frac{\partial^2}{\partial t^2}+\vec\nabla^2\right)\Psi=m^2\Psi \vert \cdot(-1)\Rightarrow \frac{\partial^2}{\partial t^2}\Psi-\vec\nabla^2\Psi=-m^2\Psi\Rightarrow$ $\partial^\mu\partial_\mu\Psi=-m^2\Psi\Rightarrow\partial^\mu\partial_\mu\Psi+m^2\Psi=0\Rightarrow \boxed{(\partial^\mu\partial_\mu+m^2)\Psi=0}$

Dirac Equation

Start like Klein Gord	$E^2=p^2+m^2\Rightarrow \hat H_D^2=\hat p^2+m^2=(-i\vec\nabla)^2+m^2=-\vec\nabla^2+m^2$ … (1) Schrödinger: $i\partial_t\Psi=\hat H_D\Psi \vert i\partial_t\cdot \Rightarrow -\partial_t^2\Psi=i\partial_t(\hat H_D\Psi)\Rightarrow$ $-\partial_t^2\Psi=i(\partial_t\hat H_D)\Psi+i\hat H_D\partial_t\Psi\vert\partial_t\hat H_D=0\Rightarrow -\partial_t^2\Psi=\hat H_D i\partial_t\Psi \vert i\partial_t\Psi=\hat H_D\Psi\Rightarrow -\partial_t^2\Psi=\hat H_D^2\Psi\overset{(1)}{\Rightarrow}-\partial_t^2\Psi=(-\vec\nabla^2+m^2)\Psi$ … (2)	
Ansatz	$\Psi(\vec r,t)=\begin{pmatrix}\Psi_1(\vec r,t)\\ \Psi_2(\vec r,t)\\ \Psi_3(\vec r,t)\\ \Psi_4(\vec r,t)\end{pmatrix}$ (spinor ∈ $\mathbb{C}^4$)	In order to allow Lorentz-covariance, $\hat H_D$, in x-space, must be linear in spatial derivatives: $\hat H_D=\underline{\alpha}_i\hat p_i+\underline\beta m$ … (3) with $\underline\alpha_i$ and $\underline\beta$ being 4x4 matrices acting on the components of Ψ
Derivation of Dirac Equation	$(1)\Rightarrow \hat H_D^2=-\vec\nabla^2+m^2\Rightarrow \hat H_D^2=-\partial_i\partial_i+m^2\Rightarrow \hat H_D^2=-\partial_i\partial_j\delta_{ij}+m^2\overset{(3)}{\Rightarrow}(\underline\alpha_i\hat p_i+\underline\beta m)(\underline\alpha_j\hat p_j+\underline\beta m)=-\partial_i\partial_j\delta_{ij}+m^2\Rightarrow$ $\frac1i\underline\alpha_i\partial_i\frac1i\underline\alpha_j\partial_j+\underline\beta m\underline\beta m+\underline\beta m\frac1i\underbrace{\underline\alpha_j\partial_j}_{j\to i}+\frac1i\underline\alpha_i\partial_i\underline\beta m=(-\partial_i\partial_j\delta_{ij}+m^2)\mathbb 1$ $-\underline\alpha_i\underline\alpha_j\partial_i\partial_j+m^2\underline\beta^2+\frac1i m\underline\beta\underline\alpha_i\partial_i+\frac1i m\underline\alpha_i\underline\beta\partial_i=(-\partial_i\partial_j\delta_{ij}+m^2)\mathbb1$ $-\underline\alpha_i\underline\alpha_j\partial_i\partial_j+m^2\underline\beta^2-im\left(\underline\beta\underline\alpha_i+\underline\alpha_i\underline\beta\right)\partial_i=(-\partial_i\partial_j\delta_{ij}+m^2)\mathbb1$ Coefficients of -$\partial_i\partial_j$: $\underline\alpha_i\underline\alpha_j=\delta_{ij}\mathbb1 \begin{cases}\overset{if\ i=j}{\Longrightarrow}\underline\alpha_i\underline\alpha_i=\frac12[\underline\alpha_i,\underline\alpha_i]_+=\mathbb1\\ \overset{if\ i\neq j}{\Longrightarrow}\underline\alpha_i\underline\alpha_j=\mathbb0=\underline\alpha_j\underline\alpha_i\Rightarrow\frac12[\underline\alpha_i,\underline\alpha_j]_+=\mathbb0\end{cases}\Rightarrow\boxed{[\underline\alpha_i,\underline\alpha_j]_+=2\delta_{ij}\mathbb1}$ Clifford algebra Coefficients of m^2: $\boxed{\underline\beta^2=\mathbb1}$ solved by $\boxed{\underline\beta=\gamma^0=\begin{pmatrix}\mathbb1_2&\mathbb0_2\\ \mathbb0_2&-\mathbb1_2\end{pmatrix}}$ Coefficients of $\mathbb0$: $\underline\alpha_i\underline\beta+\underline\beta\underline\alpha_i=[\underline\alpha_i,\underline\beta]_+=\mathbb0$ solved by $\boxed{\alpha_i=\begin{pmatrix}\mathbb0_2&\sigma_i\\ \sigma_i&\mathbb0_2\end{pmatrix}}$ with $\boxed{\sigma_1=\begin{pmatrix}0&1\\ 1&0\end{pmatrix},\sigma_2=\begin{pmatrix}0&-i\\ i&0\end{pmatrix},\sigma_3=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}}$	
Free particle Dirac Equation with γ Matrices	$i\partial_t\Psi=\hat H_D\Psi\overset{(3)}{\Rightarrow}i\partial_t\Psi=(\underline\alpha_i\hat p_i+\underline\beta m)\Psi\vert\hat p_i=\frac1i\partial_i\Rightarrow i\partial_t\Psi=\left(\frac1i\underline\alpha_i\partial_i+\underline\beta m\right)\Psi\Rightarrow i\partial_t\Psi-\frac1i\underline\alpha_i\partial_i\Psi-\underline\beta m\Psi=0\Rightarrow$ $i\partial_t\Psi+i\underline\alpha_i\partial_i\Psi-\underline\beta m\Psi=0\vert\underline\beta\cdot\Rightarrow i\underline\beta\partial_t\Psi+i\underline\beta\underline\alpha_i\partial_i\Psi-\underline\beta^2m\Psi=0\vert \boxed{\underline\beta\overset{def}{=}\gamma^0,\underline\beta\underline\alpha_i=\gamma^i,\underline\beta^2=\mathbb1}$ $i\gamma^0\partial_0\Psi+i\gamma^i\partial_i\Psi-\mathbb1m\Psi=0\vert\gamma^\mu=(\gamma^0,\gamma^i)^T\Rightarrow\boxed{(i\gamma^\mu\partial_\mu-m)\Psi=0}\Rightarrow\boxed{(i\not\partial-m)\Psi=0 \text{ with } \not\partial\overset{def}{=}\gamma^\mu\partial_\mu,\ m \text{ ... rest mass}}$	

Properties of Gamma Matrices γ^μ

Gamma Matrices (Dirac representation)	$\gamma^0\overset{def}{=}\underline\beta=\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1\end{pmatrix}$	$\gamma^1\overset{def}{=}\underline\beta\underline\alpha_1=\begin{pmatrix}0&0&0&1\\ 0&0&1&0\\ 0&-1&0&0\\ -1&0&0&0\end{pmatrix}$	$\gamma^2\overset{def}{=}\underline\beta\underline\alpha_2=\begin{pmatrix}0&0&0&-i\\ 0&0&i&0\\ 0&i&0&0\\ -i&0&0&0\end{pmatrix}$	Clifford algebra
	$\gamma^3\overset{def}{=}\underline\beta\underline\alpha_3=\begin{pmatrix}0&0&1&0\\ 0&0&0&-1\\ -1&0&0&0\\ 0&1&0&0\end{pmatrix}$	$\gamma^5\overset{def}{=}i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}$	$(\gamma^0)^2=\underline\beta^2=\mathbb1$ $(\gamma^i)^2=-\mathbb1$ $\gamma^{0\dagger}=(\gamma^0)^{-1}=\gamma^0$ $(\gamma^i)^\dagger=-\gamma^i$	$[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}\mathbb1$ $(\gamma^5)^2=\mathbb1$ $\gamma^{5\dagger}=(\gamma^5)^{-1}=\gamma^5$ $\gamma^5\gamma^\mu=-\gamma^\mu\gamma^5$ $(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0$ $\gamma^0\gamma^1=-\gamma^1\gamma^0$ $\gamma^0\gamma^2=-\gamma^2\gamma^0$ $\gamma^0\gamma^3=-\gamma^3\gamma^0$
Proof $(\gamma^i)^2=-\mathbb1$	$(\gamma^i)^2=\underline\beta\underline\alpha_i\underline\beta\underline\alpha_i=\left(\underline\beta\underline\alpha_i+\underline\alpha_i\underline\beta-\underline\alpha_i\underline\beta\right)\underline\beta\underline\alpha_i=\left([\underline\beta,\underline\alpha_i]_+-\underline\alpha_i\underline\beta\right)\underline\beta\underline\alpha_k \vert [\underline\beta,\underline\alpha_i]_+=\mathbb0\Rightarrow$ $(\gamma^i)^2=-\underline\alpha_i\underline\beta\underline\beta\underline\alpha_i\vert\underline\beta\underline\beta=\underline\beta^2=\mathbb1\Rightarrow(\gamma^i)^2=-\underline\alpha_i\underline\alpha_i\vert\underline\alpha_i\underline\alpha_i=\mathbb1\Rightarrow\boxed{(\gamma^i)^2=-\mathbb1}$		Proof γ^0 hermitian	$(\gamma^0)^\dagger=\underline\beta^\dagger=\underline\beta\Rightarrow$ $(\gamma^0)^\dagger=\gamma^0$
Proof γ^i anti hermitian	$(\gamma^i)^\dagger=\left(\underline\beta\underline\alpha_i\right)^\dagger=\underline\alpha_i^\dagger\underline\beta^\dagger=\underline\alpha_i\underline\beta=\underline\alpha_i\underline\beta+\underline\beta\underline\alpha_i-\underline\beta\underline\alpha_i=[\underline\alpha_i,\underline\beta]_+-\underline\beta\underline\alpha_i\vert[\underline\alpha_i,\underline\beta]_+=\mathbb0\Rightarrow\boxed{\underline\alpha_i\underline\beta=-\underline\beta\underline\alpha_i}\ \boxed{(\gamma^i)^\dagger=-\underline\beta\underline\alpha_i=-\gamma^i}$			
$\underline\alpha_i{}^2,\underline\beta^2$	$\underline\alpha_i{}^2=\underline\beta^2=\mathbb1\Rightarrow$ Eigenvalues = ±1	$(\gamma^i)^2=-\mathbb1\Rightarrow$ Eigenvalues = ±i	$\mathrm{tr}(\gamma^\mu\gamma^\nu)=4g^{\mu\nu}$	
tracelessness	$\underline\alpha_i=\mathbb1\underline\alpha_i=\underline\beta\underline\beta\underline\alpha_i=-\underline\beta\underline\alpha_i\underline\beta\Rightarrow \mathrm{tr}(\underline\alpha_i)=\mathrm{tr}\left(-\underline\beta\underline\alpha_i\underline\beta\right)=\mathrm{tr}\left(-\underline\alpha_i\underline\beta\underline\beta\right)=\mathrm{tr}(-\underline\alpha_i)\Rightarrow\boxed{\mathrm{tr}(\underline\alpha_i)=0}\ \boxed{\mathrm{tr}\left(\underline\beta\right)=0}\ \boxed{\mathrm{tr}(\gamma^\mu)=0}$			
Repr. with Pauli matr.:	$\underline\alpha_i=\begin{pmatrix}\mathbb0_2&\sigma_i\\ \sigma_i&\mathbb0_2\end{pmatrix},\underline\beta=\begin{pmatrix}\mathbb1_2&\mathbb0_2\\ \mathbb0_2&-\mathbb1_2\end{pmatrix},\gamma^i=\underline\beta\underline\alpha_i=\begin{pmatrix}\mathbb0_2&\sigma_i\\ -\sigma_i&\mathbb0_2\end{pmatrix}$			

Probabiliy Density of Dirac Equation

Adjoint Dirac equation	$(i\gamma^\mu\partial_\mu-m)\Psi=0\vert^\dagger\Rightarrow\Psi^\dagger\left(-i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu-m\right)=0\vert(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0\Rightarrow\Psi^\dagger\left(-i(\gamma^0\gamma^\mu\gamma^0)\overleftarrow\partial_\mu-m\gamma^0\gamma^0\right)=0\Rightarrow$ $\Psi^\dagger\gamma^0\left(-i\gamma^\mu\overleftarrow\partial_\mu-m\right)\cancel{\gamma^0}=0\vert\Psi^\dagger\gamma^0\overset{def}{=}\overline\Psi\Rightarrow\overline\Psi\left(-i\gamma^\mu\overleftarrow\partial_\mu-m\right)=0\vert\cdot(-1)\Rightarrow\overline\Psi\left(i\gamma^\mu\overleftarrow\partial_\mu+m\right)=0\vert\gamma^\mu\overleftarrow\partial_\mu\overset{def}{=}\overleftarrow{\not\partial}$ $\boxed{\overline\Psi\left(i\overleftarrow{\not\partial}+m\right)=0 \text{ with } \overleftarrow{\not\partial}\overset{def}{=}\gamma^\mu\overleftarrow\partial_\mu \text{ and } \overline\Psi\overset{def}{=}\Psi^\dagger\gamma^0}$	
continuity equation	Dirac: $(i\not\partial-m)\Psi=0\vert\overline\Psi\cdot\Rightarrow\overline\Psi(i\not\partial-m)\Psi=0$ … (1), adjoint Dirac: $\overline\Psi(i\overleftarrow{\not\partial}+m)=0\vert\cdot\Psi\Rightarrow\overline\Psi\left(i\overleftarrow{\not\partial}+m\right)\Psi=0$ … (2) (2) + (1) $\Rightarrow i\overline\Psi\overleftarrow{\not\partial}\Psi+\cancel{\overline\Psi m\Psi}+i\overline\Psi\not\partial\Psi-\cancel{\overline\Psi m\Psi}=0\vert:i\Rightarrow\overline\Psi(\overleftarrow{\not\partial}+\not\partial)\Psi=0\Rightarrow\overline\Psi(\gamma^\mu\overleftarrow\partial_\mu+\gamma^\mu\partial_\mu)\Psi=0\Rightarrow$ $(\overline\Psi\gamma^\mu)\overleftarrow\partial_\mu\Psi+\overline\Psi\gamma^\mu\partial_\mu\Psi=0\vert(\overline\Psi\gamma^\mu)\overleftarrow\partial_\mu=\gamma^\mu\overline\Psi\overleftarrow\partial_\mu=\partial_\mu\overline\Psi\gamma^\mu\Rightarrow\partial_\mu\overline\Psi\gamma^\mu\Psi+\overline\Psi\gamma^\mu\partial_\mu\Psi=0\Rightarrow\boxed{\partial_\mu(\overline\Psi\gamma^\mu\Psi)=\partial_\mu j^\mu=0}$	
Probability density	$\rho=j^0=\overline\Psi\gamma^0\Psi=\Psi^\dagger\gamma^0\gamma^0\Psi=\Psi^\dagger\mathbb1\Psi=\Psi^\dagger\Psi\Rightarrow\boxed{\rho=\sum_{\alpha=1}^4\Psi_\alpha^\dagger\Psi_\alpha\ge0}$ … positive definite	$\boxed{j^\mu=\overline\Psi\gamma^\mu\Psi=\Psi^\dagger\gamma^0\gamma^\mu\Psi}$

Covariant Solutions of Dirac Equation

Free Particle Plane Wave Solution to Dirac Equation	Ansatz: Free particle, plane wave: $\Psi = \mathrm{U}(E,\vec{p})\,e^{i(\vec{p}\cdot\vec{x}-Et)} = \mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu}$ ... (1) with $\mathrm{U}(E,\vec{p})$ being a 4-component spinor $(i\gamma^\mu\partial_\mu - m)\Psi = 0 \overset{(1)}{\Longrightarrow} (i\gamma^\mu\partial_\mu - m)\left(\mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu}\right) = 0 \Longrightarrow i\gamma^\mu\partial_\mu\left(\mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu}\right) - m\,\mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu} = 0 \Longrightarrow$ $i\gamma^\mu\,\mathrm{U}(E,\vec{p})\,\partial_\mu e^{-ip^\nu x_\nu} - m\,\mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu} = 0 \Longrightarrow i\gamma^\mu\,\mathrm{U}(E,\vec{p})\,(-ip^\nu)\partial_\mu x_\nu e^{-ip^\nu x_\nu} - m\,\mathrm{U}(E,\vec{p})\,e^{-ip^\nu x_\nu} = 0 \Longrightarrow$ $\gamma^\mu\,\mathrm{U}(E,\vec{p})\,p^\nu\eta_{\mu\nu}\cancel{e^{-ip^\nu x_\nu}} - m\,\mathrm{U}(E,\vec{p})\,\cancel{e^{-ip^\nu x_\nu}} = 0 \Longrightarrow \gamma^\mu p_\mu\,\mathrm{U}(E,\vec{p}) - m\,\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow \boxed{(\gamma^\mu p_\mu - m)\,\mathrm{U}(E,\vec{p}) = 0}$ ... (2)
Special Solution for Particle at Rest	$p_\mu = (E,\vec{0}) \overset{(2)}{\Rightarrow} (\gamma^0 E - m)\,\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow \gamma^0 E\mathrm{U} = m\mathrm{U} \Longrightarrow \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}E\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} = m\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} \Longrightarrow \begin{pmatrix}Eu_1\\Eu_2\\-Eu_3\\-Eu_4\end{pmatrix} = \begin{pmatrix}mu_1\\mu_2\\mu_3\\mu_4\end{pmatrix}$ $\underbrace{\mathrm{U}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \mathrm{U}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E=m}, \underbrace{\mathrm{U}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \mathrm{U}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E=-m} \Longrightarrow \boxed{\Psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt}, \Psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}, \Psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{imt}, \Psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{imt}}$
General Free Particle Plane Wave Solution	$p_\mu = (E,-\vec{p}) \overset{(2)}{\Rightarrow} (\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\,\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow \left(\underline{\underline{\beta}}E - \underline{\underline{\beta}}\alpha_x p_x - \underline{\underline{\beta}}\alpha_y p_y - \underline{\underline{\beta}}\alpha_z p_z - m\right)\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_x\\\sigma_x&\mathbb{0}\end{pmatrix}p_x - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_y\\\sigma_y&\mathbb{0}\end{pmatrix}p_y - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_z\\\sigma_z&\mathbb{0}\end{pmatrix}p_z - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E - \begin{pmatrix}\mathbb{0}&\sigma_x\\-\sigma_x&\mathbb{0}\end{pmatrix}p_x - \begin{pmatrix}\mathbb{0}&\sigma_y\\-\sigma_y&\mathbb{0}\end{pmatrix}p_y - \begin{pmatrix}\mathbb{0}&\sigma_z\\-\sigma_z&\mathbb{0}\end{pmatrix}p_z - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\sigma_x p_x\\\sigma_x p_x&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{0}&-\sigma_y p_y\\\sigma_y p_y&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{0}&-\sigma_z p_z\\\sigma_z p_z&\mathbb{0}\end{pmatrix} - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\sigma_x p_x - \sigma_y p_y - \sigma_z p_z\\\sigma_x p_x + \sigma_y p_y + \sigma_z p_z&\mathbb{0}\end{pmatrix} - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&\mathbb{0}\end{pmatrix} - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{U}(E,\vec{p}) = 0 \Longrightarrow \begin{pmatrix}E-m&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&-E-m\end{pmatrix}\mathrm{U} = 0 \;\Big\vert\; U \overset{\text{def}}{=} \begin{pmatrix}U_A\\U_B\end{pmatrix} \Longrightarrow$ $\begin{pmatrix}E-m&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&-(E+m)\end{pmatrix}\begin{pmatrix}U_A\\U_B\end{pmatrix} = 0 \Longrightarrow \begin{array}{l}(E-m)U_A - (\vec{\sigma}\cdot\vec{p})U_B = 0\\(\vec{\sigma}\cdot\vec{p})U_A - (E+m)U_B = 0\end{array} \Longrightarrow \left.\begin{array}{l}U_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}U_B\\U_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}U_A\end{array}\right\}$ ... (3) $\vec{\sigma}\cdot\vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix}0&1\\1&0\end{pmatrix}p_x + \begin{pmatrix}0&-i\\i&0\end{pmatrix}p_y + \begin{pmatrix}1&0\\0&-1\end{pmatrix}p_z = \begin{pmatrix}p_z&p_x - ip_y\\p_x + ip_y&-p_z\end{pmatrix} \overset{(3)}{\Rightarrow}$ $U_A = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}U_B$ ... (4a), $U_B = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}U_A$ ... (4b)
	... Ansatz $U_A = \begin{pmatrix}1\\0\end{pmatrix} \overset{(4b)}{\Longrightarrow} U_B = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix} \Longrightarrow U_1 \overset{\text{def}}{=} N_1\begin{pmatrix}U_A\\U_B\end{pmatrix} = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}$ ... Ansatz $U_A = \begin{pmatrix}0\\1\end{pmatrix} \overset{(4b)}{\Longrightarrow} U_B = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix} \Longrightarrow U_2 \overset{\text{def}}{=} N_2\begin{pmatrix}U_A\\U_B\end{pmatrix} = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}$ solutions for $E > 0$ (particles) ... Ansatz $U_B = \begin{pmatrix}1\\0\end{pmatrix} \overset{(4a)}{\Longrightarrow} U_A = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix} \Longrightarrow U_3 \overset{\text{def}}{=} N_3\begin{pmatrix}U_A\\U_B\end{pmatrix} = N_3\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix}$ ... Ansatz $U_B = \begin{pmatrix}0\\1\end{pmatrix} \overset{(4a)}{\Longrightarrow} U_A = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix} \Longrightarrow U_4 \overset{\text{def}}{=} N_4\begin{pmatrix}U_A\\U_B\end{pmatrix} = N_4\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}$ solutions for $E < 0$ $(\text{anti} - \text{particles})$
Normali-zation	$U_1^\dagger U_1 \overset{!}{=} 2E \Longrightarrow \lvert N_1\rvert^2\left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2+p_y^2}{(E+m)^2}\right) = \lvert N_1\rvert^2\frac{(E+m)^2+p_z^2+p_x^2+p_y^2}{(E+m)^2} = \lvert N_1\rvert^2\frac{E^2+2Em+m^2+\vec{p}^2}{(E+m)^2} \overset{!}{=} 2E \;\Big\vert\; \vec{p}^2 = E^2 - m^2 \Longrightarrow$ $\lvert N_1\rvert^2\frac{E^2+2Em+m^2+E^2-m^2}{(E+m)^2} = \lvert N_1\rvert^2\frac{2E^2+2Em}{(E+m)^2} = \lvert N_1\rvert^2\frac{2E(E+m)}{(E+m)^2} = \lvert N_1\rvert^2\frac{2E}{E+m} \overset{!}{=} 2E \Longrightarrow \boxed{N_1 = \sqrt{E+m} = N_2}$
Used Particle Solutions	From above, we only use U_1 and U_2 (the particle solutions): $U_1 = \sqrt{E+m}\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}$ and $U_2 = \sqrt{E+m}\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}$
Dirac Sea	Historic interpretation: In the empty vacuum all negative energy states are filled up (Dirac-sea). Holes in the Dirac sea (e.g. created by photons) are anti-particles.
Feynmann-Stückelberg	Modern interpretation: Solutions with negative energy can be seen as particles with negative energy moving backwards in time. This corresponds to an anti-particle moving forward in time. $time \uparrow \underbrace{\downarrow e^-}_{E<0} \equiv \underbrace{\uparrow e^+}_{E>0}$

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Anti-Particle Solutions

Motivation	In principle, we can calculate anti-particles with spinors U_3 and U_4. But we need to use the negative value of the physical energy. Also, because U_3 and U_4 are propagating backwards in time, the momentum is the negative physical momentum. It is more convenient to write spinors in terms of physical momentum and physical energy $E = +\left\vert\sqrt{\vec{p}^2 + m^2}\right\vert$
Free Anti-Particle Plane Wave Solution to Dirac Equation	$(1) \Rightarrow (i\gamma^\mu\partial_\mu - m)\Psi = 0 \,\vert$ New ansatz, reversing signs of E and $\vec{p}$: $\Psi = \mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)}$ $(i\gamma^\mu\partial_\mu - m)\left(\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)}\right) = 0 \Longrightarrow i\gamma^\mu\partial_\mu\left(\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)}\right) - m\,\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)} = 0 \Longrightarrow$ $i\gamma^\mu\,\mathrm{V}(E,\vec{p})\,\partial_\mu e^{-i(\vec{p}\cdot\vec{x}-Et)} - m\,\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)} = 0 \Longrightarrow$ $(i\gamma^0\partial_t + i\gamma^1\partial_x + i\gamma^2\partial_y + i\gamma^3\partial_z)\left(\mathrm{V}(E,\vec{p})\, e^{-ip_xx-ip_yy-ip_zz+iEt}\right) - m\mathrm{U}e^{-i(\vec{p}\cdot\vec{x}-Et)} = 0 \Longrightarrow$ $(i\gamma^0 iE - i\gamma^1 ip_x - i\gamma^2 ip_y - i\gamma^3 ip_z)\left(\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)}\right) - m\,\mathrm{V}(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{x}-Et)} = 0 \Longrightarrow$ $(-\gamma^0\mathrm{E} + \gamma^1p_x + \gamma^2p_y + \gamma^3p_z - m)\,\mathrm{V}(E,\vec{p}) = 0 \,\vert \cdot (-1) \Longrightarrow$ $(\gamma^0\mathrm{E} - \gamma^1p_x - \gamma^2p_y - \gamma^3p_z + m)\,\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow \boxed{(\gamma^\mu p_\mu + m)\,\mathrm{V}(E,\vec{p}) = 0}$ ... (5)
General Free Particle Plane Wave Solution	$p_\mu = \begin{pmatrix}E\\-\vec{p}\end{pmatrix} \overset{(5)}{\Rightarrow} (\gamma^0E - \gamma^1p_x - \gamma^2p_y - \gamma^3p_z + m)\,\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow \left(\underline{\underline{\beta}}E - \underline{\underline{\beta}}\alpha_xp_x - \underline{\underline{\beta}}\alpha_yp_y - \underline{\underline{\beta}}\alpha_zp_z + m\right)\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_x\\\sigma_x&\mathbb{0}\end{pmatrix}p_x - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_y\\\sigma_y&\mathbb{0}\end{pmatrix}p_y - \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{0}&\sigma_z\\\sigma_z&\mathbb{0}\end{pmatrix}p_z + \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E - \begin{pmatrix}\mathbb{0}&\sigma_x\\-\sigma_x&\mathbb{0}\end{pmatrix}p_x - \begin{pmatrix}\mathbb{0}&\sigma_y\\-\sigma_y&\mathbb{0}\end{pmatrix}p_y - \begin{pmatrix}\mathbb{0}&\sigma_z\\-\sigma_z&\mathbb{0}\end{pmatrix}p_z + \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\sigma_xp_x\\\sigma_xp_x&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{0}&-\sigma_yp_y\\\sigma_yp_y&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{0}&-\sigma_zp_z\\\sigma_zp_z&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\sigma_xp_x-\sigma_yp_y-\sigma_zp_z\\\sigma_xp_x+\sigma_yp_y+\sigma_zp_z&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow$ $\left[\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}E + \begin{pmatrix}\mathbb{0}&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&\mathbb{0}\end{pmatrix} + \begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&\mathbb{1}\end{pmatrix}m\right]\mathrm{V}(E,\vec{p}) = 0 \Longrightarrow \begin{pmatrix}E+m&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&-E+m\end{pmatrix}\mathrm{V} = 0 \,\vert\, V \overset{\text{def}}{=} \begin{pmatrix}V_A\\V_B\end{pmatrix} \Longrightarrow$ $\begin{pmatrix}E+m&-\vec{\sigma}\cdot\vec{p}\\\vec{\sigma}\cdot\vec{p}&-(E-m)\end{pmatrix}\begin{pmatrix}V_A\\V_B\end{pmatrix} = 0 \Longrightarrow \begin{matrix}(E+m)V_A - (\vec{\sigma}\cdot\vec{p})V_B = 0\\(\vec{\sigma}\cdot\vec{p})V_A - (E-m)V_B = 0\end{matrix} \Longrightarrow \left.\begin{matrix}V_A = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}V_B\\V_B = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}V_A\end{matrix}\right\}$ ... (6) $\vec{\sigma}\cdot\vec{p} = \sigma_xp_x + \sigma_yp_y + \sigma_zp_z = \begin{pmatrix}0&1\\1&0\end{pmatrix}p_x + \begin{pmatrix}0&-i\\i&0\end{pmatrix}p_y + \begin{pmatrix}1&0\\0&-1\end{pmatrix}p_z = \begin{pmatrix}p_z&p_x-ip_y\\p_x+ip_y&-p_z\end{pmatrix} \overset{(6)}{\Longrightarrow}$ $V_A = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}V_B$ ... (7a), $V_B = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}V_A$ ... (7b)
	... Ansatz $V_B = \begin{pmatrix}0\\1\end{pmatrix} \overset{(7a)}{\Longrightarrow} V_A = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix} \Longrightarrow V_1 \overset{\text{def}}{=} N_1\begin{pmatrix}V_A\\V_B\end{pmatrix} = N_1\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix}$ ... Ansatz $V_B = \begin{pmatrix}1\\0\end{pmatrix} \overset{(7a)}{\Longrightarrow} V_A = \begin{pmatrix}\frac{p_z}{E+m}&\frac{p_x-ip_y}{E+m}\\\frac{p_x+ip_y}{E+m}&\frac{-p_z}{E+m}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix} \Longrightarrow V_2 \overset{\text{def}}{=} N_2\begin{pmatrix}V_A\\V_B\end{pmatrix} = N_2\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}$ } solutions for $E > 0$ (anti − particles) ... Ansatz $V_A = \begin{pmatrix}1\\0\end{pmatrix} \overset{(7b)}{\Longrightarrow} V_B = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix} \Longrightarrow V_3 \overset{\text{def}}{=} N_3\begin{pmatrix}V_A\\V_B\end{pmatrix} = N_3\begin{pmatrix}1\\0\\\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix}$ ... Ansatz $V_A = \begin{pmatrix}0\\1\end{pmatrix} \overset{(7b)}{\Longrightarrow} V_B = \begin{pmatrix}\frac{p_z}{E-m}&\frac{p_x-ip_y}{E-m}\\\frac{p_x+ip_y}{E-m}&\frac{-p_z}{E-m}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix} \Longrightarrow V_4 \overset{\text{def}}{=} N_4\begin{pmatrix}V_A\\V_B\end{pmatrix} = N_4\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix}$ } solutions for $E < 0$ (particles)
Used Anti-Particle Solutions	We use only V_1 and V_2 (anti-particle solutions): $V_1 = \sqrt{E+m}\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix}$ and $V_2 = \sqrt{E+m}\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}$

Spin and Dirac Equation

Motivation	In non-relativistic QM the Hamiltonian of a free particle commutes with the angular momentum operator: $[\hat{H},\hat{L}]=0$. In relativistic QM it does <u>not</u> commute: $[\hat{H}_D,\hat{L}]=-i\vec{\alpha}\times\vec{p}\Longrightarrow$ Angular momentum $\vec{L}$ is not conserved!
Spin	Ansatz: We introduce a new operator $\hat{S}$ $\hat{S}=\frac{1}{2}\hat{\Sigma}=\frac{1}{2}\begin{pmatrix}\vec{\sigma} & \vec{0}\\ \vec{0} & \vec{\sigma}\end{pmatrix}\Longrightarrow\hat{S}_x=\frac{1}{2}\begin{pmatrix}\sigma_x & \mathbb{0}\\ \mathbb{0} & \sigma_x\end{pmatrix},\hat{S}_y=\frac{1}{2}\begin{pmatrix}\sigma_y & \mathbb{0}\\ \mathbb{0} & \sigma_y\end{pmatrix},\hat{S}_z=\frac{1}{2}\begin{pmatrix}\sigma_z & \mathbb{0}\\ \mathbb{0} & \sigma_z\end{pmatrix}\Longrightarrow[\hat{H}_D,\hat{S}]=i\vec{\alpha}\times\vec{p}$
Tot ang mom	$\hat{J}=\hat{L}+\hat{S}\Longrightarrow[\hat{H}_D,\hat{J}]=[\hat{H}_D,\hat{L}+\hat{S}]=[\hat{H}_D,\hat{L}]+[\hat{H}_D,\hat{S}]=-i\vec{\alpha}\times\vec{p}+i\vec{\alpha}\times\vec{p}=0$
Total Spin	$\hat{S}^2=\hat{S}_x^2+\hat{S}_y^2+\hat{S}_z^2=\frac{1}{4}\begin{pmatrix}\sigma_x & \mathbb{0}\\ \mathbb{0} & \sigma_x\end{pmatrix}^2+\frac{1}{4}\begin{pmatrix}\sigma_y & \mathbb{0}\\ \mathbb{0} & \sigma_y\end{pmatrix}^2+\frac{1}{4}\begin{pmatrix}\sigma_z & \mathbb{0}\\ \mathbb{0} & \sigma_z\end{pmatrix}^2=\frac{1}{4}\left(\begin{pmatrix}\sigma_x^2 & \mathbb{0}\\ \mathbb{0} & \sigma_x^2\end{pmatrix}+\begin{pmatrix}\sigma_y^2 & \mathbb{0}\\ \mathbb{0} & \sigma_y^2\end{pmatrix}+\begin{pmatrix}\sigma_z^2 & \mathbb{0}\\ \mathbb{0} & \sigma_z^2\end{pmatrix}\right)$ $\hat{S}^2=\frac{1}{4}\left(\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mathbb{1}\end{pmatrix}+\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mathbb{1}\end{pmatrix}+\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mathbb{1}\end{pmatrix}\right)=\frac{3}{4}\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mathbb{1}\end{pmatrix}$ Angular Momentum Algebra: $\hat{S}^2\Psi=S(S+1)\Psi=\frac{3}{4}\Psi\Longrightarrow S=\frac{1}{2}$ Dirac particles have an intrinsic Spin $S=\frac{1}{2}$

Charge Conjugation

Motivation	The effect of charge conjugation is to replace particles with the corresponding antiparticles and vice versa.
Charge Conjugation Operator $\hat{C}$	In classical electro dynamics: $\begin{pmatrix}E\\ -\vec{p}\end{pmatrix}^T\to\begin{pmatrix}E-q\phi\\ -(\vec{p}-q\vec{A})\end{pmatrix}^T\Leftrightarrow p_\mu\to p_\mu-qA_\mu$ (minimal substitution). Corresponding quantum substitution: $\begin{pmatrix}\hat{E}\\ -\hat{\vec{p}}\end{pmatrix}^T=\begin{pmatrix}i\partial_t\\ -(-i\vec{\nabla})\end{pmatrix}^T\to\begin{pmatrix}i\partial_t-q\phi\\ -(-i\vec{\nabla}-q\vec{A})\end{pmatrix}^T\Leftrightarrow i\partial_\mu\to i\partial_\mu-qA_\mu\ldots(8)$ $(1)\Longrightarrow(i\gamma^\mu\partial_\mu-m)\Psi=\gamma^\mu i\partial_\mu\Psi-m\Psi=0\overset{(8)}{\Longrightarrow}\gamma^\mu(i\partial_\mu-qA_\mu)\Psi-m\Psi=i\gamma^\mu\partial_\mu\Psi-q\gamma^\mu A_\mu\Psi-m\Psi=0\vert\cdot i$ $-\gamma^\mu\partial_\mu\Psi-iq\gamma^\mu A_\mu\Psi-im\Psi=0\vert\cdot(-1)\Longrightarrow\gamma^\mu\partial_\mu\Psi+iq\gamma^\mu A_\mu\Psi+im\Psi=0\vert q\overset{\text{def}}{=}-e\Longrightarrow$ $\gamma^\mu\partial_\mu\Psi-ie\gamma^\mu A_\mu\Psi+im\Psi=0\Longrightarrow\boxed{\gamma^\mu(\partial_\mu-ieA_\mu)\Psi+im\Psi=0}\vert^*\Longrightarrow(\gamma^\mu)^*(\partial_\mu+ieA_\mu)\Psi^*-im\Psi^*=0\vert(-i\gamma^2)\cdot$ $-i\gamma^2(\gamma^\mu)^*(\partial_\mu+ieA_\mu)\Psi^*-\gamma^2m\Psi^*=0\Longrightarrow$ $-i\gamma^2[(\gamma^0)^*(\partial_0+ieA_0)+(\gamma^1)^*(\partial_1+ieA_1)+(\gamma^2)^*(\partial_2+ieA_2)+(\gamma^3)^*(\partial_3+ieA_3)]\Psi^*-\gamma^2m\Psi^*=0\ldots(9)$ $(\gamma^0)^*=\gamma^0,(\gamma^1)^*=\gamma^1,(\gamma^3)^*=\gamma^3$, but $(\gamma^2)^*=-\gamma^2\overset{(9)}{\Longrightarrow}$ $-i\gamma^2[\gamma^0(\partial_0+ieA_0)+\gamma^1(\partial_1+ieA_1)-\gamma^2(\partial_2+ieA_2)+\gamma^3(\partial_3+ieA_3)]\Psi^*-\gamma^2m\Psi^*=0\Longrightarrow$ $-i[\gamma^2\gamma^0(\partial_0+ieA_0)+\gamma^2\gamma^1(\partial_1+ieA_1)-\gamma^2\gamma^2(\partial_2+ieA_2)+\gamma^2\gamma^3(\partial_3+ieA_3)]\Psi^*-\gamma^2m\Psi^*=0\vert\gamma^2\gamma^\mu=-\gamma^\mu\gamma^2\Longrightarrow$ $-i[-\gamma^0\gamma^2(\partial_0+ieA_0)-\gamma^1\gamma^2(\partial_1+ieA_1)-\gamma^2\gamma^2(\partial_2+ieA_2)-\gamma^3\gamma^2(\partial_3+ieA_3)]\Psi^*-\gamma^2m\Psi^*=0\Longrightarrow$ $-i[-\gamma^0(\partial_0+ieA_0)-\gamma^1(\partial_1+ieA_1)-\gamma^2(\partial_2+ieA_2)-\gamma^3(\partial_3+ieA_3)]\gamma^2\Psi^*-m\gamma^2\Psi^*=0\Longrightarrow$ $[\gamma^0(\partial_0+ieA_0)+\gamma^1(\partial_1+ieA_1)+\gamma^2(\partial_2+ieA_2)+\gamma^3(\partial_3+ieA_3)]i\gamma^2\Psi^*+im\,i\gamma^2\Psi^*=0\Longrightarrow$ $\gamma^\mu(\partial_\mu+ieA_\mu)i\gamma^2\Psi^*+im\,i\gamma^2\Psi^*=0\Longrightarrow\boxed{\gamma^\mu(\partial_\mu+ieA_\mu)\Psi'+im\,\Psi'=0}$ with $\boxed{\Psi'=\hat{C}\Psi=i\gamma^2\Psi^*}$
Charge Conjugation Operator coverts particle to anti-particle	$\Psi_1=N_1U_1e^{i(\vec{p}\cdot\vec{x}-Et)}=\sqrt{E+m}\begin{pmatrix}1\\ 0\\ \frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\end{pmatrix}e^{i(\vec{p}\cdot\vec{x}-Et)};\ \Psi_1'=\hat{C}\Psi_1=i\gamma^2\Psi_1^*=\sqrt{E+m}\,i\begin{pmatrix}0&0&0&-i\\ 0&0&i&0\\ 0&i&0&0\\ -i&0&0&0\end{pmatrix}\begin{pmatrix}1\\ 0\\ \frac{p_z}{E+m}\\ \frac{p_x-ip_y}{E+m}\end{pmatrix}e^{-i(\vec{p}\cdot\vec{x}-Et)}$ $\Psi_1'=\sqrt{E+m}\begin{pmatrix}0&0&0&1\\ 0&0&-1&0\\ 0&-1&0&0\\ 1&0&0&0\end{pmatrix}\begin{pmatrix}1\\ 0\\ \frac{p_z}{E+m}\\ \frac{p_x-ip_y}{E+m}\end{pmatrix}e^{-i(\vec{p}\cdot\vec{x}-Et)}=\sqrt{E+m}\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\ 0\\ 1\end{pmatrix}e^{-i(\vec{p}\cdot\vec{x}-Et)}\Longrightarrow\Psi_1'=N_1V_1e^{-i(\vec{p}\cdot\vec{x}-Et)}$
Ant.part. Op.	$\hat{H}_D^{(V)}=-i\frac{\partial}{\partial t},\hat{p}^{(V)}=+i\vec{\nabla},\hat{S}^{(V)}=-\hat{S}$

Helicity

Particles at Rest

For particles at rest, spinors $U_1(E,\vec{0})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $U_2(E,\vec{0})=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are Eigenstates of $\hat{S}_z=\frac{1}{2}\begin{pmatrix}\sigma_z & \mathbb{0}\\ \mathbb{0} & \sigma_z\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\Longrightarrow$

$$\hat{S}_zU_1(E,\vec{0})=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{1}{2}U_1(E,\vec{0});$$

$$\hat{S}_zU_2(E,\vec{0})=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=-\frac{1}{2}U_2(E,\vec{0})$$

Particles moving in z-direction

In general, spinors U_1,U_2,V_1,V_2 of moving particles are not Eigenstates of $\hat{S}_z$. But they are for particles moving in $\pm z$ direction:

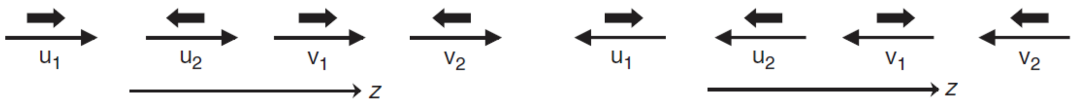
$$\hat{S}_zU_1(E,0,0,\pm p_z)=\frac{1}{2}\sqrt{E+m}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\\frac{\pm p_z}{E+m}\\0\end{pmatrix}=\frac{1}{2}\sqrt{E+m}\begin{pmatrix}1\\0\\\frac{\pm p_z}{E+m}\\0\end{pmatrix}=\frac{1}{2}U_1(E,0,0,\pm p_z)$$

$$\hat{S}_zU_2(E,0,0,\pm p_z)=\frac{1}{2}\sqrt{E+m}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\\\frac{\mp p_z}{E+m}\end{pmatrix}=-\frac{1}{2}\sqrt{E+m}\begin{pmatrix}0\\1\\0\\\frac{\mp p_z}{E+m}\end{pmatrix}=-\frac{1}{2}U_2(E,0,0,\pm p_z)$$

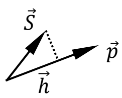
Equivalently for the anti-particles with $\hat{S}_z^{(v)}=-\hat{S}_z$:

$$\hat{S}_z^{(v)}V_1(E,0,0,\pm p_z)=-\hat{S}_zV_1(E,0,0,\pm p_z)=\frac{1}{2}V_1(E,0,0,\pm p_z),$$
$$\hat{S}_z^{(v)}V_2(E,0,0,\pm p_z)=-\hat{S}_zV_2(E,0,0,\pm p_z)=-\frac{1}{2}V_2(E,0,0,\pm p_z)$$

Hence, for particles / antiparticles with momentum $\vec{p}=p\vec{e}_z$, U_1 and V_1 represent spin up, and U_2 and V_2 represent spin down



Helicity



$\hat{S}_z$ does not produce a "good" quantum number, because it does not commute with $\hat{H}_D$: $[\hat{H}_D,\hat{S}_z]\neq 0$. However, the component of the spin along the direction of flight is a "good" quantum number: $[\hat{H}_D,\hat{S}\cdot\vec{p}]=0$

We define Helicity as normalized component of the particles spin along its direction of flight:

$$\hat{h}=\frac{\hat{S}\cdot\vec{p}}{|\vec{p}|}=\frac{1}{2}\frac{\hat{\Sigma}\cdot\vec{p}}{|\vec{p}|}=\frac{1}{2|\vec{p}|}\begin{pmatrix}\vec{\sigma}\cdot\vec{p} & \vec{0}\\ \vec{0} & \vec{\sigma}\cdot\vec{p}\end{pmatrix}$$

Helicity states are $+\frac{1}{2}$ ("right handed") and $-\frac{1}{2}$ ("left handed") (see below)
Left handed particles can participate in weak interaction
Helicity is not Lorenz-invariant, a trafo to a reference frame with opposite spin is possible

We are looking for eigenstates which are eigenstates for both $\hat{H}_D$ and $\hat{h}$:

$$\hat{h}U=\lambda U\Longrightarrow\frac{1}{2|\vec{p}|}\begin{pmatrix}\vec{\sigma}\cdot\vec{p} & \vec{0}\\ \vec{0} & \vec{\sigma}\cdot\vec{p}\end{pmatrix}\begin{pmatrix}U_A\\U_B\end{pmatrix}=\lambda\begin{pmatrix}U_A\\U_B\end{pmatrix}\ldots(8)\Longrightarrow\begin{matrix}(\vec{\sigma}\cdot\vec{p})U_A=2p\lambda U_A\ldots(8a)\\(\vec{\sigma}\cdot\vec{p})U_B=2p\lambda U_B\ldots(8b)\end{matrix}$$

$(8a)\cdot(\vec{\sigma}\cdot\vec{p})\Longrightarrow(\vec{\sigma}\cdot\vec{p})^2U_A=2p\lambda(\vec{\sigma}\cdot\vec{p})U_A|(\vec{\sigma}\cdot\vec{p})^2=\vec{p}^2\Longrightarrow\vec{p}^2U_A=2p\lambda(\vec{\sigma}\cdot\vec{p})U_A\overset{(8a)}{\Longrightarrow}\vec{p}^2U_A=2p\lambda 2p\lambda U_A=4p^2\lambda^2U_A\Longrightarrow$

$4\lambda^2=1\Longrightarrow\boxed{\lambda=\pm\frac{1}{2}}\ldots(9)$ Because the spinors corresponding to the two helicity states are also eigenstates of the Dirac equation, U_A and U_B are related by equation (3) from section "General Free Particle Solution": $U_B=\frac{\vec{\sigma}\cdot\vec{p}}{E+m}U_A\Longrightarrow$

$(\vec{\sigma}\cdot\vec{p})U_A=(E+m)U_B\overset{(8a)}{\Longrightarrow}2p\lambda\vec{U}_A=(E+m)\vec{U}_B\Longrightarrow\boxed{\vec{U}_B=\frac{2p\lambda}{E+m}\vec{U}_A}\ldots(10)$

Assumption: Particle moves in general (ϑ,φ) direction: $\vec{p}=\begin{pmatrix}p_x\\p_y\\p_z\end{pmatrix}=\begin{pmatrix}p\sin(\vartheta)\cos(\varphi)\\p\sin(\vartheta)\sin(\varphi)\\p\cos(\vartheta)\end{pmatrix}\ldots(11)$

$$\vec{\sigma}\cdot\vec{p}=\sigma_xp_x+\sigma_yp_y+\sigma_zp_z=\begin{pmatrix}0&1\\1&0\end{pmatrix}p_x+\begin{pmatrix}0&-i\\i&0\end{pmatrix}p_y+\begin{pmatrix}1&0\\0&-1\end{pmatrix}p_z=\begin{pmatrix}p_z & p_x-ip_y\\p_x+ip_y & -p_z\end{pmatrix}\overset{(11)}{\Longrightarrow}$$

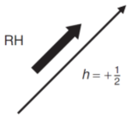
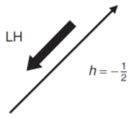
$$\vec{\sigma}\cdot\vec{p}=\begin{pmatrix}p\cos(\vartheta) & p\sin(\vartheta)\cos(\varphi)-ip\sin(\vartheta)\sin(\varphi)\\p\sin(\vartheta)\cos(\varphi)+ip\sin(\vartheta)\sin(\varphi) & -p\cos(\vartheta)\end{pmatrix}\Longrightarrow$$

$$\vec{\sigma}\cdot\vec{p}=p\begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\,(\cos(\varphi)-i\sin(\varphi))\\\sin(\vartheta)\,(\cos(\varphi)+i\sin(\varphi)) & -\cos(\vartheta)\end{pmatrix}=p\begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\,e^{-i\varphi}\\\sin(\vartheta)\,e^{i\varphi} & -\cos(\vartheta)\end{pmatrix}\overset{(8a)}{\Longrightarrow}$$

$$p\begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\,e^{-i\varphi}\\\sin(\vartheta)\,e^{i\varphi} & -\cos(\vartheta)\end{pmatrix}U_A=2p\lambda U_A\Big|\text{ansatz: }\vec{U}_A=\begin{pmatrix}a\\b\end{pmatrix}\Longrightarrow$$

$$\begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\,e^{-i\varphi}\\\sin(\vartheta)\,e^{i\varphi} & -\cos(\vartheta)\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=2\lambda\begin{pmatrix}a\\b\end{pmatrix}\Longrightarrow\begin{matrix}a\cos(\vartheta)+b\sin(\vartheta)\,e^{-i\varphi}=2\lambda a\ldots(12a)\\a\sin(\vartheta)\,e^{i\varphi}-b\cos(\vartheta)=2\lambda b\ldots(12b)\end{matrix}$$

$(12a):a\Longrightarrow\cos(\vartheta)+\frac{b}{a}\sin(\vartheta)\,e^{-i\varphi}=2\lambda\Longrightarrow\frac{b}{a}\sin(\vartheta)\,e^{-i\varphi}=2\lambda-\cos(\vartheta)\Longrightarrow\boxed{\frac{b}{a}=\frac{2\lambda-\cos(\vartheta)}{\sin(\vartheta)}e^{i\varphi}}\ldots(13)$

right-handed particle helicity spinor	right-handed particle $\Longrightarrow \lambda=+\frac{1}{2}\overset{(13)}{\Longrightarrow}\frac{b}{a}=\frac{1-\cos(\vartheta)}{\sin(\vartheta)}e^{i\varphi}\Big\vert\ \alpha\overset{\text{def}}{=}\frac{\vartheta}{2}\Longrightarrow\frac{b}{a}=\frac{1-\cos(2\alpha)}{\sin(2\alpha)}e^{i\varphi}\Big\vert\ \cos(2\alpha)=1-2\sin^2(\alpha)\Longrightarrow 1-\cos(2\alpha)=2\sin^2(\alpha)$ $\frac{b}{a}=\frac{2\sin^2(\alpha)}{\sin(2\alpha)}e^{i\varphi}\Big\vert\ \sin(2\alpha)=2\sin(\alpha)\cos(\alpha)\Longrightarrow\frac{b}{a}=\frac{2\sin^2(\alpha)}{2\sin(\alpha)\cos(\alpha)}e^{i\varphi}=\frac{\sin(\alpha)e^{i\varphi}}{\cos(\alpha)}\Longrightarrow\boxed{\frac{b}{a}=\frac{\sin\left(\frac{\vartheta}{2}\right)e^{i\varphi}}{\cos\left(\frac{\vartheta}{2}\right)}\text{ for }\lambda=+\frac{1}{2}}$ from ansatz $\vec{U}_A=\begin{pmatrix}a\\b\end{pmatrix}\Longrightarrow\vec{U}_A=N\begin{pmatrix}\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}\ldots(14a)\quad\lambda=\frac{1}{2}\overset{(10)}{\Longrightarrow}\vec{U}_B=\frac{p}{E+m}\vec{U}_A\Big\vert\overset{(14a)}{\Longrightarrow}\vec{U}_B=N\begin{pmatrix}\frac{p}{E+m}\cos\left(\frac{\vartheta}{2}\right)\\\frac{p}{E+m}e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}\ldots(14b)$ $N=\sqrt{E+m}\overset{(14ab)}{\Longrightarrow}\boxed{U_\uparrow=\sqrt{E+m}\begin{pmatrix}\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\\\frac{p}{E+m}\cos\left(\frac{\vartheta}{2}\right)\\\frac{p}{E+m}e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}}\ldots(15)$ 
left-handed particle helicity spinor	left-handed particle $\Longrightarrow\lambda=-\frac{1}{2}\overset{(13)}{\Longrightarrow}\frac{b}{a}=\frac{-1-\cos(\vartheta)}{\sin(\vartheta)}e^{i\varphi}\Big\vert\ \alpha\overset{\text{def}}{=}\frac{\vartheta}{2}\Longrightarrow\frac{b}{a}=\frac{-1-\cos(2\alpha)}{\sin(2\alpha)}e^{i\varphi}\Big\vert\ \cos(2\alpha)=1-2\sin^2(\alpha),\ \sin(2\alpha)=2\sin(\alpha)\cos(\alpha)\Longrightarrow$ $\frac{b}{a}=\frac{-1-1+2\sin^2(\alpha)}{2\sin(\alpha)\cos(\alpha)}e^{i\varphi}=\frac{-1-1+2\sin^2(\alpha)}{2\sin(\alpha)\cos(\alpha)}e^{i\varphi}=\frac{-2(1-\sin^2(\alpha))}{2\sin(\alpha)\cos(\alpha)}e^{i\varphi}=\frac{-\cos^2(\alpha)}{\sin(\alpha)\cos(\alpha)}e^{i\varphi}=\frac{-\cos(\alpha)}{\sin(\alpha)}e^{i\varphi}\Longrightarrow\boxed{\frac{b}{a}=\frac{\cos\left(\frac{\vartheta}{2}\right)e^{i\varphi}}{-\sin\left(\frac{\vartheta}{2}\right)}\text{ for }\lambda=-\frac{1}{2}}$ from ansatz $\vec{U}_A=\begin{pmatrix}a\\b\end{pmatrix}\Longrightarrow\vec{U}_A=N\begin{pmatrix}-\sin\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}\ldots(16a)\quad\lambda=-\frac{1}{2}\overset{(10)}{\Longrightarrow}\vec{U}_B=-\frac{p}{E+m}\vec{U}_A\Big\vert\overset{(16a)}{\Longrightarrow}\vec{U}_B=N\begin{pmatrix}\frac{p}{E+m}\sin\left(\frac{\vartheta}{2}\right)\\-\frac{p}{E+m}e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}\ldots(16b)$ $N=\sqrt{E+m}\overset{(16ab)}{\Longrightarrow}\boxed{U_\downarrow=\sqrt{E+m}\begin{pmatrix}-\sin\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\\\frac{p}{E+m}\sin\left(\frac{\vartheta}{2}\right)\\-\frac{p}{E+m}e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}}\ldots(17)$ 
anti-particle helicity spinors	analogous: $\boxed{V_\uparrow=\sqrt{E+m}\begin{pmatrix}\frac{p}{E+m}\sin\left(\frac{\vartheta}{2}\right)\\-\frac{p}{E+m}e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\\-\sin\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}}\ldots(18a)\quad\boxed{V_\downarrow=\sqrt{E+m}\begin{pmatrix}\frac{p}{E+m}\cos\left(\frac{\vartheta}{2}\right)\\\frac{p}{E+m}e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\\\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}}\ldots(18b)$
$E\gg m$	$\frac{p}{E+m}\to\frac{\sqrt{\vec{p}^2}}{E}\approx\frac{\sqrt{m^2+\vec{p}^2}}{E}=\frac{E}{E}=1\Longrightarrow U_\uparrow=\sqrt{E}\begin{pmatrix}\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\\\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}\quad U_\downarrow=\sqrt{E}\begin{pmatrix}-\sin\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\\\sin\left(\frac{\vartheta}{2}\right)\\-e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}\quad V_\uparrow=\sqrt{E}\begin{pmatrix}\sin\left(\frac{\vartheta}{2}\right)\\-e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\\-\sin\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right)\end{pmatrix}\quad V_\downarrow=\sqrt{E}\begin{pmatrix}\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\\\cos\left(\frac{\vartheta}{2}\right)\\e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)\end{pmatrix}$

Intrinsic Parity of Dirac Fermions

Parity Operator	Parity Operator: $x'=-x, y'=-y, z'=-z, t'=t;\quad \Psi'=\hat{P}\Psi;\quad \hat{P}\Psi'=\hat{P}\hat{P}\Psi=\Psi$ Dirac Equation with Ψ' (1) $\Longrightarrow(i\gamma^\mu\partial_\mu-m)\Psi'=0\Longrightarrow(i\gamma^0\partial_0+i\gamma^1\partial_1+i\gamma^2\partial_2+i\gamma^3\partial_3-m)\Psi'=0\ldots(19)$ $(i\gamma^0\partial_0+i\gamma^1\partial_1+i\gamma^2\partial_2+i\gamma^3\partial_3-m)\hat{P}\Psi=0\vert\hat{P}\cdot\Longrightarrow\hat{P}\left[(i\gamma^0\partial_0+i\gamma^1\partial_1+i\gamma^2\partial_2+i\gamma^3\partial_3-m)\hat{P}\Psi\right]=0\Longrightarrow$ $\hat{P}(i\gamma^0\partial_0+i\gamma^1\partial_1+i\gamma^2\partial_2+i\gamma^3\partial_3-m)\hat{P}\hat{P}\Psi=0\Longrightarrow(i\gamma^0\partial_0-i\gamma^1\partial_1-i\gamma^2\partial_2-i\gamma^3\partial_3-m)\hat{P}\Psi'=0\vert\gamma^0\cdot\Longrightarrow$ $(i\gamma^0\gamma^0\partial_0-i\gamma^0\gamma^1\partial_1-i\gamma^0\gamma^2\partial_2-i\gamma^0\gamma^3\partial_3-\gamma^0m)\hat{P}\Psi'=0\vert\gamma^0\gamma^0=\mathbb{1},\ \gamma^0\gamma^1=-\gamma^1\gamma^0,\gamma^0\gamma^2=-\gamma^2\gamma^0,\gamma^0\gamma^3=-\gamma^3\gamma^0\Longrightarrow$ $(i\gamma^0\gamma^0\partial_0+i\gamma^1\gamma^0\partial_1+i\gamma^2\gamma^0\partial_2+i\gamma^3\gamma^0\partial_3-\gamma^0m)\hat{P}\Psi'=0\Longrightarrow$ $(i\gamma^0\gamma^0\hat{P}\partial_0+i\gamma^1\gamma^0\hat{P}\partial_1+i\gamma^2\gamma^0\hat{P}\partial_2+i\gamma^3\gamma^0\hat{P}\partial_3-\gamma^0\hat{P}m)\Psi'=0\ldots(20)\Longrightarrow$ $(20)=(19)$ if $\gamma^0\hat{P}\overset{!}{=}\mathbb{1}$. compare with $\gamma^0\gamma^0=\mathbb{1}\Longrightarrow\boxed{\hat{P}=\gamma^0}$
Intrinsic parity	The intrinsic parity of a fundamental particle is defined by the action of the parity operator $\hat{P}$ on a spinor for a particle at rest. $\hat{P}U_1=\gamma^0U_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=U_1;$ analogous: $\hat{P}U_2=U_2;\ \hat{P}V_1=-V_1;\ \hat{P}V_2=-V_2$ **Intrinsic parity of particles is positive; intrinsic parity of antiparticle is negative**.

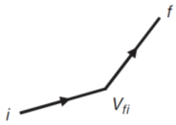
Perturbation Theory

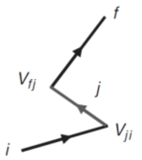
Second order perturbation theory

In deriving Fermi's Golden Rule, we assumed $c_k(t) \approx \delta_{ik}$.

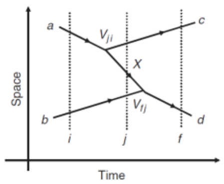
Now we assume $c_i(t) = 1, c_k(t) = \frac{1}{i\hbar} T_{ki} \int_0^t e^{i(E_k - E_i)\tau}\, d\tau \Longrightarrow$

$$T_{fi} = \underbrace{\langle f|\hat V|i\rangle}_{\text{scattering in static potential}} + \underbrace{\sum_{j\neq i} \frac{\langle f|\hat V|j\rangle\langle j|\hat V|i\rangle}{E_i - E_j}}_{\text{scattering via intermediate state } |j\rangle} + \cdots$$





scattering first possible time ordering



Particle a emits exchange particle X.
Later, X is absorbed by b.

$a + b \to c + d$

$|i\rangle = a + b$

$|j\rangle = c + b + X$

$|f\rangle = c + d$

$\vec p_X = \vec p_a - \vec p_c$ ... (A)

$$T_{fi}^{ab} = \frac{\langle f|\hat V|j\rangle\langle j|\hat V|i\rangle}{E_i - E_j} = \frac{\langle c+d|\hat V|c+b+X\rangle\langle c+b+X|\hat V|a+b\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|\hat V|b+X\rangle\langle c+X|\hat V|a\rangle}{E_i - E_j} = \frac{\langle d|\hat V|b+X\rangle\langle c+X|\hat V|a\rangle}{(E_a+E_b)-(E_b+E_c+E_X)}$$

$$\boxed{T_{fi}^{ab} = \frac{\langle d|\hat V|b+X\rangle\langle c+X|\hat V|a\rangle}{E_a - E_c - E_X} = \frac{T_{b+X\to d}\, T_{a\to c+X}}{E_a - E_c - E_X}} \text{ ... (1)}$$

Instead of T_{fi}, we want a Lorentz invariant matrix element $\mathcal{M}_{fi}$: $\boxed{\mathcal{M}_{fi} = T_{fi} \prod_k \sqrt{2E_k}}$ ... (2) with k .... index of all particles

$\overset{(2)}{\Rightarrow} T_{fi} = \mathcal{M}_{fi} \prod_k \frac{1}{\sqrt{2E_k}} \overset{(1)}{\Rightarrow}$

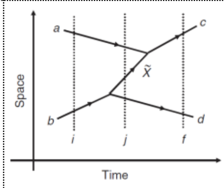
$T_{a\to c+X} = \frac{\mathcal{M}_{a\to c+X}}{\sqrt{2E_a 2E_c 2E_X}} \overset{\text{def}}{=} \frac{g_a}{\sqrt{2E_a 2E_c 2E_X}}$ ... (3a) with g_a ... coupling strength at $a \to c + X$ vertex

$T_{b+X\to d} = \frac{\mathcal{M}_{b+X\to d}}{\sqrt{2E_b 2E_d 2E_X}} \overset{\text{def}}{=} \frac{g_b}{\sqrt{2E_b 2E_d 2E_X}}$ ... (3b) with g_b ... coupling strength at $b + X \to d$ vertex

$$T_{fi}^{ab} \overset{(1)}{=} \frac{T_{b+X\to d}\, T_{a\to c+X}}{E_a - E_c - E_X} \overset{(3ab)}{=} \frac{g_a}{\sqrt{2E_a 2E_c 2E_X}} \frac{g_b}{\sqrt{2E_b 2E_d 2E_X}} \frac{1}{E_a - E_c - E_X} = \frac{1}{2E_X\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{g_a g_b}{E_a - E_c - E_X} \overset{(2)}{\Rightarrow}$$

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{g_a g_b}{E_a - E_c - E_X} \sqrt{2E_a 2E_b 2E_c 2E_d} \Longrightarrow \boxed{\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X} \frac{g_a g_b}{E_a - E_c - E_X}} \text{ ... (4)}$$

scattering second possible time ordering



Particle b emits exchange particle $\tilde X$.
Later, $\tilde X$ is absorbed by a.

$a + b \to c + d$

$|i\rangle = a + b$

$|j\rangle = a + b + \tilde X$

$|f\rangle = c + d$

$\vec p_{\tilde X} + \vec p_a = \vec p_c \Longrightarrow \vec p_{\tilde X} = \vec p_c - \vec p_a$ ... (B)

$$T_{fi}^{ba} = \frac{\langle f|\hat V|j\rangle\langle j|\hat V|i\rangle}{E_i - E_j} = \frac{\langle c+d|\hat V|a+d+\tilde X\rangle\langle a+d+\tilde X|\hat V|a+b\rangle}{E_i - E_j}$$

$$T_{fi}^{ba} = \frac{\langle c|\hat V|a+\tilde X\rangle\langle d+\tilde X|\hat V|b\rangle}{E_i - E_j} = \frac{\langle c|\hat V|a+\tilde X\rangle\langle d+\tilde X|\hat V|b\rangle}{(E_a+E_b)-(E_a+E_d+E_X)}$$

$$\boxed{T_{fi}^{ba} = \frac{\langle c|\hat V|a+\tilde X\rangle\langle d+\tilde X|\hat V|b\rangle}{E_b - E_d - E_X} = \frac{T_{a+\tilde X\to c}\, T_{b\to d+\tilde X}}{E_b - E_d - E_X}} \text{ ... (5)}$$

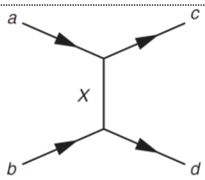
$T_{b\to d+\tilde X} = \frac{\mathcal{M}_{b\to d+\tilde X}}{\sqrt{2E_b 2E_d 2E_X}} \overset{\text{def}}{=} \frac{g_b}{\sqrt{2E_b 2E_d 2E_X}}$ ... (6a) with g_b ... coupling strength at $b \to d + \tilde X$ vertex

$T_{a+\tilde X\to c} = \frac{\mathcal{M}_{a+\tilde X\to c}}{\sqrt{2E_a 2E_c 2E_X}} \overset{\text{def}}{=} \frac{g_a}{\sqrt{2E_a 2E_c 2E_X}}$ ... (6b) with g_a ... coupling strength at $a + \tilde X \to c$ vertex

$$T_{fi}^{ba} \overset{(5)}{=} \frac{T_{a+\tilde X\to c}\, T_{b\to d+\tilde X}}{E_b - E_d - E_X} \overset{(6ab)}{=} \frac{g_b}{\sqrt{2E_b 2E_d 2E_X}} \frac{g_a}{\sqrt{2E_a 2E_c 2E_X}} \frac{1}{E_b - E_d - E_X} = \frac{1}{2E_X\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{g_a g_b}{E_b - E_d - E_X} \overset{(2)}{\Rightarrow}$$

$$\mathcal{M}_{fi}^{ba} = \frac{1}{2E_X\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{g_a g_b}{E_b - E_d - E_X} \sqrt{2E_a 2E_b 2E_c 2E_d} \Longrightarrow \boxed{\mathcal{M}_{fi}^{ba} = \frac{1}{2E_X} \frac{g_a g_b}{E_b - E_d - E_X}} \text{ ... (7)}$$

scattering process (t-channel)



$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba} \overset{(4)(7)}{\Longrightarrow}$$

$$\mathcal{M}_{fi} = \frac{1}{2E_X}\frac{g_a g_b}{E_a - E_c - E_X} + \frac{1}{2E_X}\frac{g_a g_b}{E_b - E_d - E_X} = \frac{g_a g_b}{2E_X}\left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X}\right) \Bigg| \; E_a + E_b = E_c + E_d \Longrightarrow E_b - E_d = E_c - E_a$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_X}\left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_c - E_a - E_X}\right) = \frac{g_a g_b}{2E_X}\left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X}\right)$$

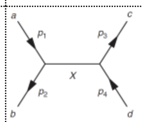
$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_X} \frac{E_a - E_c + E_X - (E_a - E_c - E_X)}{(E_a - E_c - E_X)(E_a - E_c + E_X)} = \frac{g_a g_b}{2E_X} \frac{2E_X}{E_a^2 - 2E_aE_c + E_c^2 - E_X^2} \Longrightarrow$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2} = \frac{g_a g_b}{(E_a - E_c)^2 - \vec p_X^2 - m_X^2} \;\Bigg|\; \left.\begin{array}{l} (A) \Longrightarrow \vec p_X = \vec p_a - \vec p_c \Longrightarrow \vec p_X^2 = (\vec p_a - \vec p_c)^2 \\ (B) \Longrightarrow \vec p_{\tilde X} = \vec p_c - \vec p_a \Longrightarrow \vec p_{\tilde X}^2 = (\vec p_c - \vec p_a)^2 = (\vec p_a - \vec p_c)^2 \end{array}\right\} \text{in both orders: } \vec p_X^2 = (\vec p_a - \vec p_c)^2$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec p_a - \vec p_c)^2 - m_X^2} \;\Bigg|\; q^\mu = q_a^\mu - q_c^\mu = \begin{pmatrix} E_a \\ \vec p_a \end{pmatrix} - \begin{pmatrix} E_c \\ \vec p_c \end{pmatrix} = \begin{pmatrix} E_a - E_c \\ \vec p_a - \vec p_c \end{pmatrix} \quad q^\mu q_\mu < 0 \text{ ...space-like}$$

Propagator: $\boxed{\mathcal{M}_{fi} = \frac{g_a g_b}{q^\mu q_\mu - m_X^2} \text{ with } q^\mu = q_a^\mu - q_c^\mu = q_d^\mu - q_b^\mu}$ ... (8) E and $\vec p$ conserved in every vertex. X not on mass shell

annihilation process (s-channel)



Propagator: $\boxed{\mathcal{M}_{fi} = \frac{g_a g_b}{q^\mu q_\mu - m_X^2} \text{ with } q^\mu = p_1^\mu + p_2^\mu = q_3^\mu + q_4^\mu}$ ... (9) $q^\mu q_\mu > 0$ ...time-like

QED

charged particle in EM-field

Charged particle moving in EM field: Minimal coupling to potential.

Classical minimal substitution: $\vec{p} \to \vec{p} - q\vec{A}, E \to E - q\phi$ QM: $\boxed{i\partial_\mu \to i\partial_\mu - qA_\mu}$ ... (1) with $A_\mu = (\phi, -\vec{A})$

Dirac equation: $(i\gamma^\mu\partial_\mu - m)\Psi = 0 \overset{(1)}{\Longrightarrow} (\gamma^\mu(i\partial_\mu - qA_\mu) - m)\Psi = 0 \Longrightarrow i\gamma^\mu\partial_\mu\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = 0 \Longrightarrow$

$i\gamma^0\partial_0\Psi + i\gamma^1\partial_1\Psi + i\gamma^2\partial_2\Psi + i\gamma^3\partial_3\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = i\gamma^0\partial_t\Psi + i\vec{\gamma}\vec{\nabla}\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = 0 \,|\, i\partial_t = \hat{H} \Longrightarrow$

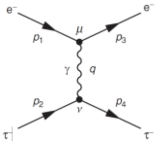
$\gamma^0\hat{H}\Psi + i\vec{\gamma}\vec{\nabla}\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = 0 \Longrightarrow \gamma^0\hat{H}\Psi = m\Psi - i\vec{\gamma}\vec{\nabla}\Psi + q\gamma^\mu A_\mu\Psi \,|\, \gamma^0\cdot \Longrightarrow$

$\gamma^0\gamma^0\hat{H}\Psi = \gamma^0 m\Psi - i\gamma^0\vec{\gamma}\vec{\nabla}\Psi + q\gamma^0\gamma^\mu A_\mu\Psi \,|\, \gamma^0\gamma^0 = \mathbb{1} \Longrightarrow \hat{H}\Psi = (\gamma^0 m - i\gamma^0\vec{\gamma}\vec{\nabla})\Psi + q\gamma^0\gamma^\mu A_\mu\Psi \,|\, \gamma^0 = \underline{\beta}, \vec{\gamma} = \underline{\beta}\vec{\alpha} \Longrightarrow$

$\hat{H}\Psi = (\underline{\beta}m - i\underline{\beta}\underline{\beta}\vec{\alpha}\vec{\nabla})\Psi + q\gamma^0\gamma^\mu A_\mu\Psi \,|\, \underline{\beta}\underline{\beta} = \mathbb{1}, \hat{\vec{p}} = -i\vec{\nabla} \Longrightarrow \hat{H}\Psi = \underbrace{(\underline{\beta}m + \vec{\alpha}\hat{\vec{p}})}_{\hat{H}_D}\Psi + \underbrace{q\gamma^0\gamma^\mu A_\mu\Psi}_{pot.energy}$

Potential energy of a spin ½ particle in an EM field $\boxed{V_D = q\gamma^0\gamma^\mu A_\mu}$ ... (10)

Polarization of photon in $e^-\tau^-$ scattering



interaction at $e^-\gamma$ vertex: $\langle\Psi_3|\hat{V}_D|\Psi_1\rangle \overset{(10)}{=} \langle\Psi_3|q\gamma^0\gamma^\mu A_\mu|\Psi_1\rangle \overset{q=q_e}{=} U_e^\dagger(p_3^\sigma)\, q_e\gamma^0\gamma^\mu\varepsilon_\mu\, U_e(p_1^\sigma)$ ... (11a)

interaction at $\tau^-\gamma$ vertex: $\langle\Psi_4|\hat{V}_D|\Psi_2\rangle \overset{(10)}{=} \langle\Psi_4|q\gamma^0\gamma^\mu A_\mu|\Psi_2\rangle \overset{q=q_\tau}{=} -U_\tau^\dagger(p_4^\sigma)\, q_\tau\gamma^0\gamma^\nu\varepsilon_\nu\, U_\tau(p_2^\sigma)$ ... (11b)

propagator: $\sum_\lambda \frac{\varepsilon_\mu^{(\lambda)}\varepsilon_\mu^{*(\lambda)}}{q^\alpha q_\alpha - m_\gamma^2} \overset{m_\gamma=0}{=} \sum_\lambda \frac{\varepsilon_\mu^{(\lambda)}\varepsilon_\mu^{*(\lambda)}}{q^\alpha q_\alpha} = -\frac{g_{\mu\nu}}{q^\alpha q_\alpha}$ ... (11c) with polarization $\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \ldots, \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$\mathcal{M} = \langle\Psi_3|\hat{V}_D|\Psi_1\rangle \sum_\lambda \frac{\varepsilon_\mu^{(\lambda)}\varepsilon_\mu^{*(\lambda)}}{q^\alpha q_\alpha - m_\gamma^2}\langle\Psi_4|\hat{V}_D|\Psi_2\rangle \overset{(11abc)}{=} [U_e^\dagger(p_3^\sigma)\, q_e\gamma^0\gamma^\mu\, U_e(p_1^\sigma)]\frac{-g_{\mu\nu}}{q^\alpha q_\alpha}[U_\tau^\dagger(p_4^\sigma)\, q_\tau\gamma^0\gamma^\nu\, U_\tau(p_2^\sigma)] \Longrightarrow$

$\mathcal{M} = -[q_e\, U_e^\dagger(p_3^\sigma)\,\gamma^0\gamma^\mu\, U_e(p_1^\sigma)]\frac{g_{\mu\nu}}{q^\alpha q_\alpha}[q_\tau\, U_\tau^\dagger(p_4^\sigma)\,\gamma^0\gamma^\nu\, U_\tau(p_2^\sigma)] \,\Big|\, \overline{U} \overset{\text{def}}{\equiv} U^\dagger\gamma^0 \Longrightarrow$

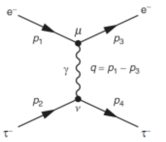
$\mathcal{M} = -[q_e\, \overline{U}_e(p_3^\sigma)\,\gamma^\mu\, U_e(p_1^\sigma)]\frac{g_{\mu\nu}}{q^\alpha q_\alpha}[q_\tau\, \overline{U}_\tau(p_4^\sigma)\,\gamma^\nu\, U_\tau(p_2^\sigma)]$ with 4-vector currents: $\left\{\boxed{\begin{array}{l} j_e^\mu = \overline{U}_e(p_3^\sigma)\,\gamma^\mu\, U_e(p_1^\sigma) \\ j_\tau^\mu = \overline{U}_\tau(p_4^\sigma)\,\gamma^\mu\, U_\tau(p_2^\sigma)\end{array}}\right\} \Longrightarrow$

$\mathcal{M} = -q_e q_\tau \frac{j_e^\mu j_\tau^\nu g_{\mu\nu}}{q^\alpha q_\alpha} \Longrightarrow \boxed{\mathcal{M} = -q_e q_\tau \frac{j_e^\mu j_\mu^\tau}{q^\alpha q_\alpha}}$ ... (12)

Feynman Rules for QED

$-i\mathcal{M} = \ldots$						
	initial state particle:	$U(p^\sigma)$	initial state antiparticle:	$\overline{V}(p^\sigma)$	initial state photon:	$\varepsilon_\mu(p^\sigma)$
	final state particle:	$\overline{U}(p^\sigma)$	final state antiparticle:	$V(p^\sigma)$	final state photon:	$\varepsilon_\nu^*(p^\sigma)$
	photon propagator:	$-\frac{ig_{\mu\nu}}{q^\alpha q_\alpha}$	fermion propagator:	$-\frac{i(\gamma^\mu q_\mu + m)}{q^\alpha q_\alpha - m^2}$	QED vertex:	$-iq\gamma^\mu$

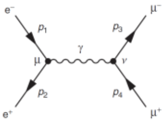
example $e^-\tau^-$ scattering



$\underbrace{-i\mathcal{M}^{LO}}_{\text{"LO"... "leading order"}} = \underbrace{\overbrace{[\overline{U}_e(p_3^\sigma)}^{\text{arrow out: LS adjoint}}_{\text{final } e^-}\underbrace{(-iq_e\gamma^\mu)}_{\mu-\text{vertex}}\overbrace{U_e(p_1^\sigma)]}^{\text{arrow in: RS spinor}}_{\text{initial } e^-}}_{\text{interaction of } e^- \text{ with photon } \gamma}\underbrace{\frac{-ig_{\mu\nu}}{q^\alpha q_\alpha}}_{\text{photon propagator}}\underbrace{\overbrace{[\overline{U}_\tau(p_4^\sigma)}^{\text{arrow out: LS adjoint}}_{\text{final } \tau^-}\underbrace{(-iq_\tau\gamma^\nu)}_{\nu-\text{vertex}}\overbrace{U_\tau(p_2^\sigma)]}^{\text{arrow in: RS spinor}}_{\text{initial } \tau^-}}_{\text{interaction of } \tau^- \text{ with photon } \gamma}$

Particles where the arrow points "in" get the spinor on the right side (RS), particles where the arrow points "out" get the adjoint spinor on the left side (LS)

example $e^+e^-\to\mu^+\mu^-$ annihilation



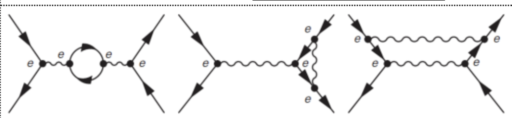
$\underbrace{-i\mathcal{M}^{LO}}_{\text{"LO"... "leading order"}} = \underbrace{\overbrace{[\overline{V}_{e+}(p_2^\sigma)}^{\text{arrow out: LS adjoint}}_{\text{initial } e^+}(-ie\gamma^\mu)\overbrace{U_{e-}(p_1^\sigma)]}^{\text{arrow in: RS spinor}}_{\text{initial } e^-}}_{e^-e^+\to\gamma}\underbrace{\frac{-ig_{\mu\nu}}{q^\alpha q_\alpha}}_{\text{photon propagator}}\underbrace{\overbrace{[\overline{U}_{\mu-}(p_3^\sigma)}^{\text{arrow out: LS adjoint}}_{\text{final } \mu^-}\underbrace{(-ie\gamma^\nu)}_{\nu-\text{vertex}}\overbrace{V_{\mu+}(p_4^\sigma)]}^{\text{arrow in: RS spinor}}_{\text{final } \mu^+}}_{\gamma\to\mu^+\mu^-}$ $|(-i)(-i) = -1 \Longrightarrow$

$-i\mathcal{M}^{LO} = -[\overline{V}_{e+}(p_2^\sigma)\,(e\gamma^\mu)\, U_{e-}(p_1^\sigma)]\frac{-ig_{\mu\nu}}{q^\alpha q_\alpha}[\overline{U}_{\mu-}(p_3^\sigma)\,(e\gamma^\nu)\, V_{\mu+}(p_4^\sigma)] \,\Big|\, \cdot\frac{1}{-i} \Longrightarrow$

$\mathcal{M}^{LO} = -[\overline{V}_{e+}(p_2^\sigma)\,(e\gamma^\mu)\, U_{e-}(p_1^\sigma)]\frac{g_{\mu\nu}}{q^\alpha q_\alpha}[\overline{U}_{\mu-}(p_3^\sigma)\,(e\gamma^\nu)\, V_{\mu+}(p_4^\sigma)] = -\frac{e^2}{q^\alpha q_\alpha}g_{\mu\nu}[\overline{V}_{e+}(p_2^\sigma)\,\gamma^\mu\, U_{e-}(p_1^\sigma)][\overline{U}_{\mu-}(p_3^\sigma)\,\gamma^\nu\, V_{\mu+}(p_4^\sigma)]$

$\mathcal{M}^{LO} = -\frac{e^2}{q^\alpha q_\alpha}g_{\mu\nu}j_e^\mu j_{mu}^\nu \Longrightarrow \boxed{\mathcal{M}^{LO} = -\frac{e^2}{q^\alpha q_\alpha}j_e^\nu j_\nu^{mu}}$ with $\boxed{j_e^\mu = \overline{V}_{e+}(p_2^\sigma)\,\gamma^\mu\, U_{e-}(p_1^\sigma)}$ and $\boxed{j_{mu}^\nu = \overline{U}_{\mu-}(p_3^\sigma)\,\gamma^\nu\, V_{\mu+}(p_4^\sigma)}$

Higher order $e^+e^-\to\mu^+\mu^-$ annihilation



For each of the next leading order (NLO) diagrams with four vertices (3 examples on the left) the matrix element $\mathcal{M}_i^{NLO}$ must be determined. Then, the total matrix element is:

$\mathcal{M}_{fi} = \mathcal{M}^{LO} + \sum_i \mathcal{M}_i^{NLO} + \sum_j \mathcal{M}_j^{NNLO} + \cdots$

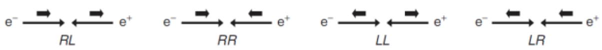
Because $\alpha_{EM} = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137}$ and $\mathcal{M}^{LO} \propto e^2 \propto$, $\mathcal{M}^{NLO} \propto e^4 \propto \alpha_{EM}^2, \mathcal{M}^{NNLO} \propto e^6 \propto \alpha_{EM}^3, \ldots$ there is rapid convergence.

Physical observables depend on $|\mathcal{M}_{fi}|^2 = (\mathcal{M}^{LO} + \sum_i \mathcal{M}_i^{NLO} + \sum_j \mathcal{M}_j^{NNLO} + \cdots)(\mathcal{M}^{LO^*} + \sum_i \mathcal{M}_i^{NLO^*} + \sum_j \mathcal{M}_j^{NNLO^*} + \cdots)$

Spin and Cross Section in $e^-e^+ \to \mu^+\mu^-$ Annihilation

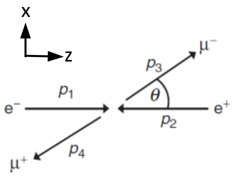
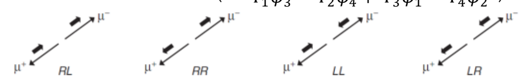
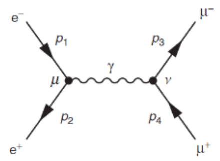
Spin Sums

Four possible helicity configurations in the initial state (fat arrow: spin, thin arrow: direction of motion)



$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{LL}|^2) \Longrightarrow$

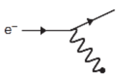
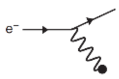
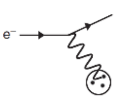
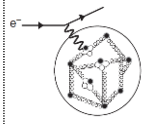
$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}(|\mathcal{M}_{RR\to RR}|^2 + |\mathcal{M}_{RR\to RL}|^2 + |\mathcal{M}_{RR\to LR}|^2 + |\mathcal{M}_{RR\to LL}|^2 + |\mathcal{M}_{RL\to RR}|^2 + |\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{RL\to LR}|^2 + |\mathcal{M}_{RL\to LL}|^2 +$
$|\mathcal{M}_{LR\to RR}|^2 + |\mathcal{M}_{LR\to RL}|^2 + |\mathcal{M}_{LR\to LR}|^2 + |\mathcal{M}_{LR\to LL}|^2 + |\mathcal{M}_{LL\to RR}|^2 + |\mathcal{M}_{LL\to RL}|^2 + |\mathcal{M}_{LL\to LR}|^2 + |\mathcal{M}_{LL\to LL}|^2)$

Ultra-Relativistic Limit $E \gg \sqrt{s}$	 $p_1^\mu \overset{c=1}{=} \begin{pmatrix} m_e + E_{kin} \\ 0 \\ 0 \\ p_1 \end{pmatrix}$ $\sqrt{s} = E_{kin} \gg m_e \Longrightarrow p_1^\mu \overset{c=1}{=} \begin{pmatrix} E \\ 0 \\ 0 \\ p_1 \end{pmatrix}$ $E \overset{c=1}{=} \sqrt{m^2 + \vec{p}^2} \overset{\vec{p}^2 \gg m}{\approx} p \Longrightarrow$ $p_1^\mu \overset{c=1}{=} \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}, p_2^\mu \overset{c=1}{=} \begin{pmatrix} E \\ 0 \\ 0 \\ -E \end{pmatrix}, p_3^\mu \overset{c=1}{=} \begin{pmatrix} E \\ E\sin(\vartheta) \\ 0 \\ E\cos(\vartheta) \end{pmatrix}, p_4^\mu \overset{c=1}{=} \begin{pmatrix} E \\ -E\sin(\vartheta) \\ 0 \\ -E\cos(\vartheta) \end{pmatrix}$ ... (1)
Initial state spinors $E \gg \sqrt{s}$	$E \gg m_e \Longrightarrow U_\uparrow = \sqrt{E}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \; U_\downarrow = \sqrt{E}\begin{pmatrix} -\sin\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \; V_\uparrow = \sqrt{E}\begin{pmatrix} \sin\left(\frac{\vartheta}{2}\right) \\ -e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \\ -\sin\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \; V_\downarrow = \sqrt{E}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}$ $\left.\begin{matrix} \vartheta_1 = 0 \\ \varphi_1 \overset{!}{=} 0 \end{matrix}\right\} \Longrightarrow U_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, U_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ $\left.\begin{matrix} \vartheta_2 = \pi \\ \varphi_2 \overset{!}{=} \pi \end{matrix}\right\} \Longrightarrow V_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, V_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$
Final state spinors $E \gg \sqrt{s}$	$\left.\begin{matrix} \vartheta_3 = \vartheta \\ \varphi_3 \overset{!}{=} 0 \end{matrix}\right\} \Longrightarrow U_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}, U_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}$ $\left.\begin{matrix} \vartheta_4 = \pi - \vartheta \\ \varphi_4 \overset{!}{=} \pi \end{matrix}\right\} \Longrightarrow V_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} \sin\left(\frac{\pi-\vartheta}{2}\right) \\ \cos\left(\frac{\pi-\vartheta}{2}\right) \\ -\sin\left(\frac{\pi-\vartheta}{2}\right) \\ -\cos\left(\frac{\pi-\vartheta}{2}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -\cos\left(\frac{\vartheta}{2}\right) \\ -\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \; V_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} \cos\left(\frac{\pi-\vartheta}{2}\right) \\ -\sin\left(\frac{\pi-\vartheta}{2}\right) \\ \cos\left(\frac{\pi-\vartheta}{2}\right) \\ -\sin\left(\frac{\pi-\vartheta}{2}\right) \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\vartheta}{2}\right) \\ -\cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}$
Muon and Electron 4-Currents	$j^\mu = \overline{\Psi}\gamma^\mu\phi = \Psi^\dagger\gamma^0\gamma^\mu\phi = \begin{pmatrix} \Psi_1^*\phi_1 + \Psi_2^*\phi_2 + \Psi_3^*\phi_3 + \Psi_4^*\phi_4 \\ \Psi_1^*\phi_4 + \Psi_2^*\phi_3 + \Psi_3^*\phi_2 + \Psi_4^*\phi_1 \\ -i(\Psi_1^*\phi_4 - \Psi_2^*\phi_3 + \Psi_3^*\phi_2 - \Psi_4^*\phi_1) \\ \Psi_1^*\phi_3 - \Psi_2^*\phi_4 + \Psi_3^*\phi_1 - \Psi_4^*\phi_2 \end{pmatrix}$   Muon current: $j_{mu}^\nu = \bar{u}(p_3^\alpha)\gamma^\nu v(p_4^\alpha)$ Electron current: $j_e^\mu = \bar{v}(p_2^\alpha)\gamma^\nu u(p_1^\alpha)$
helicity combinations for e^+e^- initial state:	$j_{e,RL}^\mu = \bar{v}_\uparrow(p_2^\alpha)\gamma^\mu u_\downarrow(p_1^\alpha) = 2E(0,-1,-i,0)^T$ $j_{e,RR}^\mu = \bar{v}_\uparrow(p_2^\alpha)\gamma^\mu u_\uparrow(p_1^\alpha) = (0,0,0,0)^T$ $j_{e,LL}^\mu = \bar{v}_\downarrow(p_2^\alpha)\gamma^\mu u_\downarrow(p_1^\alpha) = (0,0,0,0)^T$ $j_{e,LR}^\mu = \bar{v}_\downarrow(p_2^\alpha)\gamma^\mu u_\uparrow(p_1^\alpha) = 2E(0,-1,i,0)^T$ helicity combinations for $\mu^+\mu^-$ final state: $j_{\mu,RL}^\nu = \bar{u}_\uparrow(p_3^\alpha)\gamma^\nu v_\downarrow(p_4^\alpha) = 2E(0,-\cos(\vartheta),i,\sin(\vartheta))^T$ $j_{\mu,RR}^\nu = \bar{u}_\uparrow(p_3^\alpha)\gamma^\nu v_\uparrow(p_4^\alpha) = (0,0,0,0)^T$ $j_{\mu,LL}^\nu = \bar{u}_\downarrow(p_3^\alpha)\gamma^\nu v_\downarrow(p_4^\alpha) = (0,0,0,0)^T$ $j_{\mu,LR}^\nu = \bar{u}_\downarrow(p_3^\alpha)\gamma^\nu v_\uparrow(p_4^\alpha) = 2E(0,-\cos(\vartheta),-i,\sin(\vartheta))^T$
$\frac{d\sigma}{d\Omega^*}$ ultra-relativistic differential cross section $e^+e^- \to \mu^+\mu^-$ annihilation	For each helicity combination: $\mathcal{M} = -\frac{e^2}{q^\alpha q_\alpha} j_e^\nu j_\nu^\mu = -\frac{e^2}{s} j_e^\nu j_\nu^{mu} = -\frac{e^2}{4E^2} j_e^\nu j_\nu^{mu}$, and for $E \gg m$: RR's, LL's are irrelevant $\Longrightarrow$ $\mathcal{M}_{RL\to RL} = -\frac{e^2}{4E^2} j_{eRL}^\nu j_\nu^{muRL} = -\frac{e^2}{4E^2} 2E2E \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ +\cos(\vartheta) \\ -i \\ -\sin(\vartheta) \end{pmatrix} = -e^2(-\cos(\vartheta) - 1) = e^2(\cos(\vartheta) + 1)$ $\mathcal{M}_{LR\to LR} = -\frac{e^2}{4E^2} j_{eLR}^\nu j_\nu^{muLR} = -\frac{e^2}{4E^2} 2E2E \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ +\cos(\vartheta) \\ +i \\ -\sin(\vartheta) \end{pmatrix} = -e^2(-\cos(\vartheta) - 1) = e^2(\cos(\vartheta) + 1)$ $\mathcal{M}_{RL\to LR} = -\frac{e^2}{4E^2} j_{eRL}^\nu j_\nu^{muLR} = -\frac{e^2}{4E^2} 2E2E \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ +\cos(\vartheta) \\ +i \\ -\sin(\vartheta) \end{pmatrix} = -e^2(-\cos(\vartheta) + 1) = e^2(\cos(\vartheta) - 1)$ $\mathcal{M}_{LR\to RL} = -\frac{e^2}{4E^2} j_{eLR}^\nu j_\nu^{muRL} = -\frac{e^2}{4E^2} 2E2E \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ +\cos(\vartheta) \\ -i \\ -\sin(\vartheta) \end{pmatrix} = -e^2(-\cos(\vartheta) + 1) = e^2(\cos(\vartheta) - 1)$ $\langle \lvert\mathcal{M}_{fi}\rvert^2 \rangle = \frac{1}{4}(\lvert\mathcal{M}_{RL\to RL}\rvert^2 + \lvert\mathcal{M}_{LR\to LR}\rvert^2 + \lvert\mathcal{M}_{RL\to LR}\rvert^2 + \lvert\mathcal{M}_{LR\to RR}\rvert^2) = \frac{1}{4}(2e^4(\cos(\vartheta)+1)^2 + 2e^4(\cos(\vartheta)-1)^2)$ $\langle \lvert\mathcal{M}_{fi}\rvert^2 \rangle = \frac{1}{4} 2e^4(\cos^2(\vartheta) + 2\cos(\vartheta) + 1 + \cos^2(\vartheta) - 2\cos(\vartheta) + 1) \Longrightarrow \boxed{\langle \lvert\mathcal{M}_{fi}\rvert^2 \rangle = 16\pi^2\alpha^2(1 + \cos^2(\vartheta))}$ ... (2) $\boxed{\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\lvert\mathcal{M}_{fi}\rvert^2}$ $p_f^* = p_f^* \overset{E \gg m}{\approx} E \Longrightarrow \frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\lvert\mathcal{M}_{fi}\rvert^2 \Longrightarrow \boxed{\frac{d\sigma}{d\Omega^*} = \frac{e^4}{64\pi^2 s}(1 + \cos^2(\vartheta)) = \frac{\alpha^2}{4s}(1 + \cos^2(\vartheta))}$
Total $e^+e^- \to \mu^+\mu^-$ cross-section	$\sigma = \int \frac{d\sigma}{d\Omega^*} d\Omega^* = \frac{\alpha^2}{4s} 2\pi \int_0^\pi (1 + \cos^2(\vartheta))\sin(\vartheta)\, d\vartheta = \frac{4\pi\alpha^2}{3s}$ Mandelstam variables: $s = 2p_1^\mu p_\mu^2;\ t = -2p_1^\mu p_\mu^3;\ u = -2p_1^\mu p_\mu^4$
Lorentz-invariant form	$p_1^\mu p_\mu^2 \overset{(1)}{=} 2E^2, p_1^\mu p_\mu^3 \overset{(1)}{=} 2E^2(1 - \cos(\vartheta)), p_1^\mu p_\mu^4 \overset{(1)}{=} 2E^2(1 + \cos(\vartheta)) \Longrightarrow \boxed{\langle \lvert\mathcal{M}_{fi}\rvert^2 \rangle = 2e^4\frac{(p_1^\mu p_\mu^3)^2 + (p_1^\mu p_\mu^4)^2}{(p_1^\mu p_\mu^2)^2} = 2e^4\frac{t^2+u^2}{s^2}}$

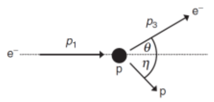
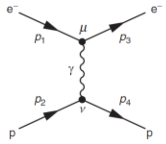
Chirality

Chiral States	The eigenstates of the γ^5 matrix are defined as left- and right-handed *chiral* states (denoted subscript R and L). In general, the solutions to the Dirac equation which are also eigenstates of γ^5 are identical to the massless helicity eigenstates $U_R = \sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}$ $U_L = \sqrt{E+m}\begin{pmatrix} -\sin\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}$ $V_R = \sqrt{E+m}\begin{pmatrix} \sin\left(\frac{\vartheta}{2}\right) \\ -e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \\ -\sin\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}$ $V_L = \sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix}$
Chiral Projection Operators	Any Dirac spinor can be decomposed into left- and right-handed chiral components: $\Psi = \Psi_R + \Psi_L = \hat{P}_R\Psi + \hat{P}_L\Psi$ $\hat{P}_R = \frac{1}{2}(\mathbb{1}+\gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, $\hat{P}_L = \frac{1}{2}(\mathbb{1}-\gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$ $\hat{P}_R U_R = U_R, \hat{P}_R U_L = 0$ $\hat{P}_R V_R = 0, \quad \hat{P}_R V_L = V_L$ $\hat{P}_L U_L = U_L, \hat{P}_L U_R = 0$ $\hat{P}_L V_L = 0, \quad \hat{P}_L V_R = V_R$
Helicity spinor $U_\uparrow$ expressed in chiral components U_R and U_L	$U_\uparrow = \hat{P}_R U_\uparrow + \hat{P}_L U_\uparrow = \frac{1}{2}(\mathbb{1}+\gamma^5)U_\uparrow + \frac{1}{2}(\mathbb{1}-\gamma^5)U_\uparrow \Longrightarrow$ $U_\uparrow = \frac{1}{2}(\mathbb{1}+\gamma^5)\sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \frac{p}{E+m}\cos\left(\frac{\vartheta}{2}\right) \\ \frac{p}{E+m}e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix} + \frac{1}{2}(\mathbb{1}-\gamma^5)\sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \frac{p}{E+m}\cos\left(\frac{\vartheta}{2}\right) \\ \frac{p}{E+m}e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix} \Longrightarrow$ $U_\uparrow = \frac{1}{2}\left(1+\frac{p}{E+m}\right)\sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right) \end{pmatrix} + \frac{1}{2}\left(1-\frac{p}{E+m}\right)\sqrt{E+m}\begin{pmatrix} -\sin\left(\frac{\vartheta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \\ \sin\left(\frac{\vartheta}{2}\right) \\ -e^{i\varphi}\cos\left(\frac{\vartheta}{2}\right) \end{pmatrix} \Longrightarrow$ $\boxed{U_\uparrow = \frac{1}{2}\left(1+\frac{p}{E+m}\right)U_R + \frac{1}{2}\left(1-\frac{p}{E+m}\right)U_L}$

Electron-Proton Scattering, General

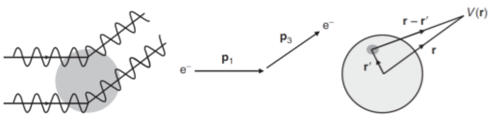
General:	The nature of $e^-p \to e^-p$ scattering depends on the wavelength of the virtual photon in comparison with the proton radius.				
$\lambda \gg r_p$ (very low energy):		Process can be described as elastic scattering of the electron in the static potential of a point-like proton.	$\lambda \sim r_p$ (higher energies):		Cross section calculation needs to account for the extended charge
$\lambda < r_p$ (high energy):		The elastic scattering cross section becomes small. The dominant process is inelastic scattering. Virtual photon interacts with constituent quark. Proton breaks up.	$\lambda \ll r_p$ (very high energies):		The wavelength of the virtual photon is sufficiently short to resolve the proton's internal structure. The proton appears to be a sea of strongly interacting quarks and gluons.

Rutherford and Mott Scattering

General	  Low energy scattering: The kinetic energy of the recoiling proton is negligible. The proton can be taken to be a fixed source of a 1/r potential. The matrix element is given by: $\mathcal{M}_{fi} = \frac{Q_q e^2}{q^\alpha q_\alpha}\left(\overline{\mathrm{U}}(p_3^\alpha)\gamma^\mu \mathrm{U}(p_1^\alpha)\right) g_{\mu\nu}\left(\overline{\mathrm{U}}(p_4^\alpha)\gamma^\nu \mathrm{U}(p_2^\alpha)\right)$
Electron Spinors	$U_\uparrow = N_e\begin{pmatrix}\cos(\vartheta/2)\\ e^{i\varphi}\sin(\vartheta/2)\\ \kappa\cos(\vartheta/2)\\ \kappa e^{i\varphi}\sin(\vartheta/2)\end{pmatrix},\ U_\downarrow = N_2\begin{pmatrix}-\sin(\vartheta/2)\\ e^{i\varphi}\cos(\vartheta/2)\\ \kappa\sin(\vartheta/2)\\ -\kappa e^{i\varphi}\cos(\vartheta/2)\end{pmatrix}$ with $N_e = \sqrt{E+m}$, $\kappa = \frac{p}{E+m_e} = \frac{\beta_e\gamma_e}{\gamma_e+1}$ If the velocity of the electron is small, then $\kappa \ll 1$ and to a good approximation the energy of the electron does not change and $\kappa(p_1^\alpha) = \kappa(p_3^\alpha)$. Taking $\varphi = 0$, we get: $\mathrm{U}_\uparrow(p_1^\alpha) = N_e\begin{pmatrix}1\\0\\ \kappa\\0\end{pmatrix}, \mathrm{U}_\downarrow(p_1^\alpha) = N_e\begin{pmatrix}0\\1\\0\\-\kappa\end{pmatrix}$ and $\mathrm{U}_\uparrow(p_3^\alpha) = N_e\begin{pmatrix}\cos(\vartheta/2)\\ \sin(\vartheta/2)\\ \kappa\cos(\vartheta/2)\\ \sin(v)\end{pmatrix}, \mathrm{U}_\downarrow(p_3^\alpha) = N_e\begin{pmatrix}-\sin(\vartheta/2)\\ \cos(\vartheta/2)\\ \kappa\sin(\vartheta/2)\\ -\kappa\cos(\vartheta/2)\end{pmatrix}$
Electron currents	$j_{e\uparrow\uparrow}^\mu = \overline{\mathrm{U}}_\uparrow(p_3^\alpha)\gamma^\mu \mathrm{U}_\uparrow(p_1^\alpha) = (E+m_e)\left((\kappa^2+1)\cos\left(\frac{\vartheta}{2}\right), 2\kappa\sin\left(\frac{\vartheta}{2}\right), +2i\kappa\sin\left(\frac{\vartheta}{2}\right), 2\kappa\cos\left(\frac{\vartheta}{2}\right)\right)^T$ $j_{e\downarrow\downarrow}^\mu = \overline{\mathrm{U}}_\downarrow(p_3^\alpha)\gamma^\mu \mathrm{U}_\downarrow(p_1^\alpha) = (E+m_e)\left((\kappa^2+1)\cos\left(\frac{\vartheta}{2}\right), 2\kappa\sin\left(\frac{\vartheta}{2}\right), -2i\kappa\sin\left(\frac{\vartheta}{2}\right), 2\kappa\cos\left(\frac{\vartheta}{2}\right)\right)^T$ $j_{e\downarrow\uparrow}^\mu = \overline{\mathrm{U}}_\uparrow(p_3^\alpha)\gamma^\mu \mathrm{U}_\downarrow(p_1^\alpha) = (E+m_e)\left((1-\kappa^2)\sin\left(\frac{\vartheta}{2}\right), 0, 0, 0\right)^T$ $j_{e\uparrow\downarrow}^\mu = \overline{\mathrm{U}}_\downarrow(p_3^\alpha)\gamma^\mu \mathrm{U}_\uparrow(p_1^\alpha) = (E+m_e)\left((\kappa^2-1)\sin\left(\frac{\vartheta}{2}\right), 0, 0, 0\right)^T$
Proton Spinors	The velocity of the recoiling proton is small ($\beta_p \ll 1$), the lower 2 components become 0, since $\kappa \ll 1$. Taking $\varphi_p = \pi$, we get: $\mathrm{U}_\uparrow(p_2^\alpha) = \sqrt{2m_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \mathrm{U}_\downarrow(p_2^\alpha) = \sqrt{2m_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and $\mathrm{U}_\uparrow(p_4^\alpha) = \sqrt{2m_p}\begin{pmatrix}\cos(\eta/2)\\ -\sin(\eta/2)\\ 0\\ 0\end{pmatrix}, \mathrm{U}_\downarrow(p_4^\alpha) = \sqrt{2m_p}\begin{pmatrix}-\sin(\eta/2)\\ -\cos(\eta/2)\\ 0\\ 0\end{pmatrix}$
P. currents	$j_{p\uparrow\uparrow}^\mu = -j_{p\downarrow\downarrow}^\mu = 2m_p\left(\cos\left(\frac{\eta}{2}\right), 0, 0, 0\right)^T$ and $j_{p\uparrow\downarrow}^\mu = -j_{p\downarrow\uparrow}^\mu = -2m_p\left(\sin\left(\frac{\eta}{2}\right), 0, 0, 0\right)^T$
Matrix element	$\mathcal{M}_{fi} = \frac{e^2}{q^\alpha q_\alpha} j_e^\mu j_\mu^p \Longrightarrow \langle\mathcal{M}_{fi}^2\rangle = \frac{1}{4}\sum\lvert\mathcal{M}_{fi}^2\rvert \Longrightarrow$ $\langle\mathcal{M}_{fi}^2\rangle = \frac{1}{4}\frac{e^4}{(q^\alpha q_\alpha)^2}4m_p^2(E+m_e)^2\left(\cos^2\left(\frac{\eta}{2}\right)+\sin^2\left(\frac{\eta}{2}\right)\right)\left(4(1+\kappa^2)^2\cos^2\left(\frac{\vartheta}{2}\right)+4(1-\kappa^2)^2\sin^2\left(\frac{\vartheta}{2}\right)\right) \Big\vert\, E = \gamma_e m_e$ $\langle\mathcal{M}_{fi}^2\rangle = \frac{4m_p^2 m_e^2 e^4(\gamma_e+1)^2}{(q^\alpha q_\alpha)^2}\left((1-\kappa^2)^2+4\kappa^2\cos^2\left(\frac{\vartheta}{2}\right)\right) \Big\vert\, \kappa = \frac{\beta_e\gamma_e}{\gamma_e+1},\ (1-\beta_e^2)\gamma_e^2 = 1 \Longrightarrow$ $\langle\mathcal{M}_{fi}^2\rangle = \frac{16m_p^2 m_e^2 e^4}{(q^\alpha q_\alpha)^2}\left(1+\beta_e^2\gamma_e^2\cos^2\left(\frac{\vartheta}{2}\right)\right)$ ... (1) In t-channel scattering process $q^\alpha q_\alpha = (q^\alpha)^2 = (p_1^\alpha - p_3^\alpha)^2$ When the recoil of the proton can be neglected, the initial and final states of the electron are $E_1 = E_3 = E, p_1 = p_3 = p \Longrightarrow$ Hence: $(q^\alpha)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -2p^\alpha p_\alpha(1-\cos(\vartheta)) = -4p^\alpha p_\alpha\sin^2\left(\frac{\vartheta}{2}\right) \overset{(1)}{\Longrightarrow}$ $\boxed{\langle\mathcal{M}_{fi}^2\rangle = \frac{m_p^2 m_e^2 e^4}{(p^\alpha p_\alpha)^2\sin^4\left(\frac{\vartheta}{2}\right)}\left(1+\beta_e^2\gamma_e^2\cos^2\left(\frac{\vartheta}{2}\right)\right)}$ ... (2)
Ruterford scattering	Electron is non-relativistic, and proton recoil can be neglected $\Longrightarrow \beta_e^2\gamma_e^2 \ll 1 \overset{(2)}{\Longrightarrow} \langle\mathcal{M}_{fi}^2\rangle = \frac{m_p^2 m_e^2 e^4}{(p^\alpha p_\alpha)^2\sin^4\left(\frac{\vartheta}{2}\right)}$ ... (3) The laboratory frame differential cross section can be written as $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p+E_1-E_1\cos(\vartheta)}\right)^2\langle\mathcal{M}_{fi}^2\rangle$ ... (4) $\Big\vert\, E_1 \approx m_e \ll m_p \Longrightarrow$ $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_p^2}\langle\mathcal{M}_{fi}^2\rangle \overset{(3)}{\Longrightarrow} \boxed{\frac{d\sigma}{d\Omega} = \frac{m_e^2 e^4}{64\pi^2(p^\alpha p_\alpha)^2\sin^4\left(\frac{\vartheta}{2}\right)}}\ E_K = \frac{p^\alpha p_\alpha}{2m_e}, e^2 = 4\pi\alpha \Longrightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2\sin^4\left(\frac{\vartheta}{2}\right)}}$
Mott scattering	Electron is relativistic, but proton recoil can still be neglected: $m_e \ll E \ll m_p \Longrightarrow \kappa \approx 1 \Longrightarrow j_{e\downarrow\uparrow}^\mu \approx 0, j_{e\uparrow\downarrow}^\mu \approx 0 \overset{(2)}{\Longrightarrow}$ $\langle\mathcal{M}_{fi}^2\rangle = \frac{m_p^2 e^4}{E^2\sin^4\left(\frac{\vartheta}{2}\right)}\cos^2\left(\frac{\vartheta}{2}\right) \xRightarrow{(4), e^2=4\pi\alpha} \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\left(\frac{\vartheta}{2}\right)}\cos^2\left(\frac{\vartheta}{2}\right)}$ .. neglecting extent of proton's charge distribution

Form Factors

General



The form factor accounts for the finite extent of the charge distribution, hence for the phase differences between contributions to the scattered wave from different points of the charge distribution. If $\lambda_\gamma \gg r_p$, then contributions are in phase, add constructively. If $\lambda_\gamma \ll r_p$, then phase stringly position dependent, negative interference strongly reduces the total amplitude when integrated.

Matrix Element, Form Factor

Potential: $V(\vec r) = \int \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} d^3r'$ … (1) In Born approximation, initial and scattered electrons are expressed as plane waves:

$\mathcal{M}_{fi} = \langle \Psi_f | V(\vec r) | \Psi_i \rangle = \int e^{-i\vec p_3\cdot\vec r}\, V(\vec r)\, e^{i\vec p_1\cdot\vec r}\, d^3r \overset{(1)}{\Rightarrow} \mathcal{M}_{fi} = \iint e^{-i\vec p_3\cdot\vec r} \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} e^{i\vec p_1\cdot\vec r}\, d^3r'\, d^3r$

$\mathcal{M}_{fi} = \iint e^{i(\vec p_1-\vec p_3)\cdot\vec r} \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} d^3r'\, d^3r \;\Big|\; \vec q = \vec p_1 - \vec p_3 \Rightarrow \mathcal{M}_{fi} = \iint e^{i\vec q\cdot\vec r} \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} d^3r'\, d^3r = \iint e^{i\vec q\cdot(\vec r-\vec r'+\vec r')} \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} d^3r'\, d^3r$

$\mathcal{M}_{fi} = \iint e^{i\vec q\cdot(\vec r-\vec r')} e^{i\vec q\cdot\vec r'} \frac{Q\,\rho(\vec r')}{4\pi|\vec r-\vec r'|} d^3r'\, d^3r \;\Big|\; \vec R \overset{\text{def}}{=} \vec r - \vec r' \Rightarrow \mathcal{M}_{fi} = \underbrace{\int e^{i\vec q\cdot\vec R} \frac{Q}{4\pi|\vec R|} d^3R}_{\text{equiv. point charge}}\; \underbrace{\int \rho(\vec r')\, e^{i\vec q\cdot\vec r'}\, d^3r'}_{\text{Form Factor } F(\vec q^2)} \Rightarrow$

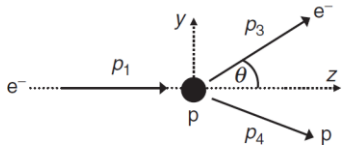
$\boxed{\mathcal{M}_{fi} = \mathcal{M}_{fi}^{pt}\, F(\vec q^2) \text{ with } F(\vec q^2) = \int \rho(\vec r')\, e^{i\vec q\cdot\vec r'}\, d^3r'}$ Form factor $F(\vec q^2)$ is a 3D Fourier transform of the charge distribution

Mott Scattering

$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\left(\frac{\vartheta}{2}\right)} \cos^2\left(\frac{\vartheta}{2}\right) |F(\vec q^2)|^2 \text{ with } F(\vec q^2) = \int \rho(\vec r')\, e^{i\vec q\cdot\vec r'}\, d^3r'}$ .. accounting for proton's charge distribution

Relativistic Electron-Proton Elastic Scattering and Rosenbluth Formula

General



At high energies, the proton's recoil cannot be neglected and the magnetic spin-spin interaction becomes important.

$p_1^\mu = \begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix}, p_2^\mu = \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix}, p_3^\mu = \begin{pmatrix} E_3 \\ 0 \\ E_3\sin(\vartheta) \\ E_3\cos(\vartheta) \end{pmatrix}, p_4^\mu = \begin{pmatrix} E_4 \\ p_4^x \\ p_4^y \\ p_4^z \end{pmatrix}$

Matrix Element

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4}\left((p_1^\mu p_\mu^2)(p_3^\nu p_\nu^4) + (p_1^\mu p_\mu^4)(p_2^\nu p_\nu^3) - m_p^2(p_1^\mu p_\mu^3) - m_e^2(p_2^\mu p_\mu^4) + 2m_p^2 m_e^2\right) \Big|\; m_e^2 \approx 0$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4}\left((p_1^\mu p_\mu^2)(p_3^\nu p_\nu^4) + (p_1^\mu p_\mu^4)(p_2^\nu p_\nu^3) - m_p^2(p_1^\mu p_\mu^3)\right) \Big|\; p_4^\mu$ not observable: $p_4^\mu = p_1^\mu + p_2^\mu - p_3^\mu$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4}\left((p_1^\mu p_\mu^2)(p_3^\nu(p_\nu^1 + p_\nu^2 - p_\nu^3)) + (p_1^\mu(p_\mu^1 + p_\mu^2 - p_\mu^3))(p_2^\nu p_\nu^3) - m_p^2(p_1^\mu p_\mu^3)\right)$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4}\left((p_1^\mu p_\mu^2)\underbrace{(p_3^\nu p_\nu^1 + p_3^\nu p_\nu^2 - p_3^\nu p_\nu^3)}_{E_1E_3(1-\cos(\vartheta))+E_3m_p-m_e^2} + \underbrace{(p_1^\mu p_\mu^1 + p_1^\mu p_\mu^2 - p_1^\mu p_\mu^3)}_{m_e^2+E_1m_p-E_1E_3(1-\cos(\vartheta))}(p_2^\nu p_\nu^3) - m_p^2(p_1^\mu p_\mu^3)\right) \Big|\; m_e^2 \approx 0$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4} m_p E_1 E_3 \left((E_1 - E_3)(1-\cos(\vartheta)) + m_p(1+\cos(\vartheta))\right)$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8e^4}{(p_1^\alpha-p_3^\alpha)^4} 2m_p E_1 E_3 \left((E_1 - E_3)\sin^2\left(\frac{\vartheta}{2}\right) + m_p\cos^2\left(\frac{\vartheta}{2}\right)\right)$ … (1)

$(p_1^\alpha - p_3^\alpha)^2 = q_\alpha q^\alpha = p_1^\alpha p_\alpha^1 + p_3^\alpha p_\alpha^3 - 2p_1^\alpha p_\alpha^3 = m_e^2 + m_e^2 - 2E_1E_3(1-\cos(\vartheta)) \approx -2E_1E_3(1-\cos(\vartheta)) = -4E_1E_3\sin^2\left(\frac{\vartheta}{2}\right) \overset{(1)}{\Rightarrow}$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{16e^4}{16E_1^2E_3^2\sin^4\left(\frac{\vartheta}{2}\right)} m_p E_1 E_3 \left((E_1 - E_3)\sin^2\left(\frac{\vartheta}{2}\right) + m_p\cos^2\left(\frac{\vartheta}{2}\right)\right) = \frac{m_p e^4}{E_1E_3\sin^4\left(\frac{\vartheta}{2}\right)}\left(\underbrace{(E_1 - E_3)}_{-q_\alpha q^\alpha/(2m_p)}\sin^2\left(\frac{\vartheta}{2}\right) + m_p\cos^2\left(\frac{\vartheta}{2}\right)\right)$

$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{m_p e^4}{E_1E_3\sin^4\left(\frac{\vartheta}{2}\right)}\left(-\frac{q_\alpha q^\alpha}{2m_p}\sin^2\left(\frac{\vartheta}{2}\right) + m_p\cos^2\left(\frac{\vartheta}{2}\right)\right) = \frac{m_p^2 e^4}{E_1E_3\sin^4\left(\frac{\vartheta}{2}\right)}\left(\cos^2\left(\frac{\vartheta}{2}\right) - \frac{q_\alpha q^\alpha}{2m_p^2}\sin^2\left(\frac{\vartheta}{2}\right)\right) \Big|\; Q^2 \overset{\text{def}}{=} -q_\alpha q^\alpha$

$\boxed{\langle|\mathcal{M}_{fi}|^2\rangle = \frac{m_p^2 e^4}{E_1E_3\sin^4\left(\frac{\vartheta}{2}\right)}\left(\cos^2\left(\frac{\vartheta}{2}\right) + \frac{Q^2}{2m_p^2}\sin^2\left(\frac{\vartheta}{2}\right)\right)}$ … (2)

Diff. cross-section

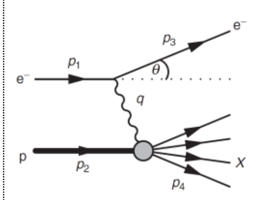
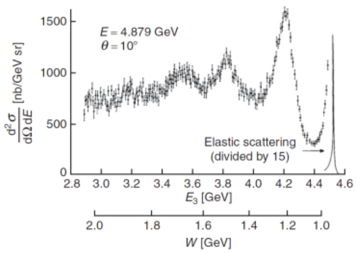
$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{m_pE_1}\right)^2 \langle|\mathcal{M}_{fi}|^2\rangle \overset{(2)}{\Rightarrow} \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\left(\frac{\vartheta}{2}\right)}\frac{E_3}{E_1}\left(\cos^2\left(\frac{\vartheta}{2}\right) + \frac{Q^2}{2m_p^2}\sin^2\left(\frac{\vartheta}{2}\right)\right) \text{ with } Q^2 \overset{\text{def}}{=} -q_\mu q^\mu}$ … (3)

Rosenbluth Formula

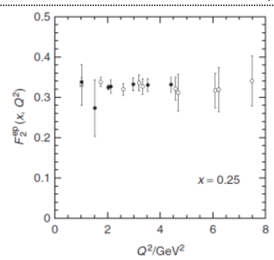
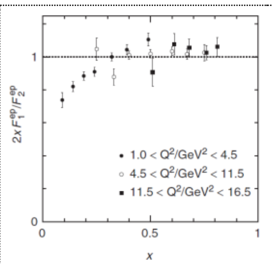
The finite size of the proton can be accounted for in (3) by introducing two form factors: $G_E(Q^2)$ for the proton's charge distribution and $G_M(Q^2)$ for the magnetic moment distribution of the proton. This leads to the most general Lorentz-invariant form for electron-proton scattering via exchange of a single proton, the Rosenbluth formula:

$\boxed{\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2\sin^4\left(\frac{\vartheta}{2}\right)}}_{\text{Rutherford}}\underbrace{\frac{E_3}{E_1}}_{\text{recoil}}\left(\underbrace{\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\left(\frac{\vartheta}{2}\right)}_{\text{em scattering } (e^- \text{ spin})} + \underbrace{2\tau G_M^2\sin^2\left(\frac{\vartheta}{2}\right)}_{\text{magnetic term } (p^+ \text{ spin})}\right) \text{ with } \tau \overset{\text{def}}{=} \frac{Q^2}{4m_p^2} = -\frac{q_\mu q^\mu}{4m_p^2}}$

Electron-Proton Inelastic Scattering, Q^2, W^2, Bjorken-x, and other Kinematic Variables

General	 Because of the finite size of the proton, elastic scattering decreases rapidly with energy. High-energy e^-p interactions are dominated by inelastic scattering processes. The hadronic final state resulting from the break-up of the proton consists of many particles. The invariant mass of the hadronic system, denoted W, depends on the four-momentum of the virtual photon, $W^2 = p_4^\mu p_\mu^4 = (p_2^\mu + q^\mu)^2$. In elastic scattering the invariant mass of the final state is the mass of the proton and it is described in terms of the electron scattering angle alone. Now we have two degrees of freedom, meaning that the kinematics must be specified by two quantities, which are usually chosen from the Lorentz-invariant quantities W^2, x, y, ν and Q^2
Q^2	$\boxed{Q^2 \overset{\text{def}}{=} -q^\mu q_\mu \geq 0}$ ... (1) $Q^2 = -(p_1^\mu - p_3^\mu)^2 = -p_1^\mu p_\mu^1 - p_3^\mu p_\mu^3 + 2p_1^\mu p_\mu^3 = -m_e^2 - m_e^2 + 2E_1E_3(1-\cos(\vartheta)) \,\vert\, m_e^2 \approx 0 \Longrightarrow$ $Q^2 \approx 2E_1E_3(1-\cos(\vartheta)) \Longrightarrow \boxed{Q^2 = 4E_1E_3\sin^2\left(\frac{\vartheta}{2}\right)}$ ... (2)
W^2	$\boxed{W^2 \overset{\text{def}}{=} p_4^\mu p_\mu^4 \geq m_p^2}$ ... (3) $W^2 = (q^\mu + p_2^\mu)^2 = q^\mu q_\mu + p_2^\mu p_\mu^2 + 2p_2^\mu q_\mu \Longrightarrow W^2 - q^\mu q_\mu - p_2^\mu p_\mu^2 = 2p_2^\mu q_\mu \overset{(1)}{\Longrightarrow}$ $W^2 + Q^2 - p_2^\mu p_\mu^2 = 2p_2^\mu q_\mu \,\vert\, p_2^\mu p_\mu^2 = m_p^2 \Longrightarrow W^2 + Q^2 - m_p^2 = 2p_2^\mu q_\mu$ ... (4)
Bjorken x ('elasticity')	$\boxed{x \overset{\text{def}}{=} \frac{Q^2}{2p_2^\mu q_\mu}}$ ... (5) $\overset{(4)}{\Longrightarrow} \boxed{x = \frac{Q^2}{Q^2+W^2-m_p^2}}$ ... (6) Because $W^2 \geq m_p^2$ and $Q^2 \geq 0 \Longrightarrow \boxed{0 \leq x \leq 1}$ When $x = 1 \Longrightarrow W^2 = m_p^2 \Longrightarrow$ elastic scattering
inelasticity y (fractional energy loss)	$\boxed{y \overset{\text{def}}{=} \frac{p_2^\mu q_\mu}{p_2^\nu p_\nu^1}}$ ... (7) $\Longrightarrow$ $\boxed{y = 1 - \frac{E_3}{E_1}}$ ... (8) In the frame where the proton is at rest: $p_1^\mu = \begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix}, p_2^\mu = \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix}, p_3^\mu = \begin{pmatrix} E_3 \\ E_3\sin(\vartheta) \\ 0 \\ E_3\cos(\vartheta) \end{pmatrix}, q^\mu = \begin{pmatrix} E_1 - E_3 \\ p_1^x - p_3^x \\ p_1^y - p_3^y \\ p_1^z - p_3^z \end{pmatrix} \Longrightarrow y = \frac{m_p(E_1-E_3)}{m_pE_1} \Longrightarrow$ y is the energy lost by the electron in the frame where the proton is at rest. In this frame, the energy of the final-state hadronic system is always greater than the energy of the initial state proton, $E_4 \geq m_p \Longrightarrow e^-$ loses energy $\Longrightarrow \boxed{0 \leq y \leq 1}$
energy loss ν	Sometimes it is more convenient to work in terms of energies rather than the fractional energy loss described in y. $\boxed{\nu \overset{\text{def}}{=} \frac{p_2^\mu q_\mu}{m_p}}$ ... (9) In the frame where the proton is at rest: $\boxed{\nu = E_1 - E_3}$ ... (10)
Relations	$x = \frac{Q^2}{2m_p\nu}$ ... (11) $x = \frac{2m_p}{s-m_p^2}\nu$ ... (12) $Q^2 = (s - m_p^2)xy$ ... (13)
Inelastic scattering at low Q^2	 For electron–proton scattering at relatively low electron energies, both elastic and inelastic scattering processes occur. Because two independent variables are required to define the kinematics of inelastic scattering, the corresponding double-differential cross section is expressed in terms of two variables, in this case $\frac{d^2\sigma}{d\Omega\, dE_3}$. For electrons detected at a fixed scattering angle, the invariant mass W of the hadronic system is linearly related to the energy E_3 of the scattered electron. Consequently the energy distribution can be interpreted in terms of W. The large peak at final-state electron energies corresponds to $W = m_p$ (elastic scattering). The peak before corresponds to resonant production of a Δ^+ baryon.

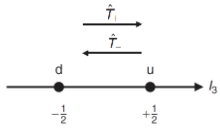
Deep inelastic scattering

Introduction	The most general Lorentz-invariant *elastic* scattering formula, the Rosenbluth formula, can be re-written using Q^2 and y: $\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\left(1 - y - \frac{m_p^2y^2}{Q^2}\right) + \frac{1}{2}y^2G_M^2\right)$ The Q^2 dependence of G_E and G_M and $\tau = \frac{Q^2}{4m_p}$ can be absorbed by $\mathrm{f}_1(Q^2)$, $\mathrm{f}_2(Q^2)$ $\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left(\left(1 - y - \frac{m_p^2y^2}{Q^2}\right)\mathrm{f}_2(Q^2) + \frac{1}{2}y^2\,\mathrm{f}_1(Q^2)\right)$ In this form: $\mathrm{f}_1(Q^2)$... magnetic $\mathrm{f}_2(Q^2)$... magnetic & electric interaction
Inelastic $ep \to eX$ scattering	$\frac{d^2\sigma}{dx\, dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left(\left(1 - y - \frac{m_p^2y^2}{Q^2}\right)\frac{\mathrm{F}_2(x,Q^2)}{x} + y^2\,\mathrm{F}_1(x,Q^2)\right)$ with "structure functions" $\underbrace{\mathrm{F}_1(x,Q^2)}_{\text{magnetic}}, \mathrm{F}_2(x,Q^2)$
Deep inelastic	$Q^2 \gg m_p^2y^2 \Longrightarrow \frac{d^2\sigma}{dx\, dQ^2} \approx \frac{4\pi\alpha^2}{Q^4}\left((1-y)\frac{\mathrm{F}_2(x,Q^2)}{x} + y^2\,\mathrm{F}_1(x,Q^2)\right)$
Bjorken scaling	 Both $\mathrm{F}_1(x,Q^2)$ and $\mathrm{F}_2(x,Q^2)$ are almost independent of Q^2. Hence, the structure functions can be written $\mathrm{F}_1(x,Q^2) \to \mathrm{F}_1(x)$ and $\mathrm{F}_2(x,Q^2) \to \mathrm{F}_2(x)$. This is strongly suggestive of scattering from point-like quarks. Callan-Cross Relation  $\mathrm{F}_2(x) = 2x\,\mathrm{F}_1(x)$

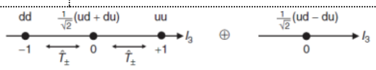
Symmetries in Quantum Mechanics

<table>
<tr><td>General:</td><td>Physical predictions must be invariant under a symmetry transformation $\Psi \to \Psi' = \hat{U}\Psi \Longrightarrow$
$\langle\Psi|\Psi\rangle = \langle\Psi'|\Psi'\rangle = \langle\hat U\Psi|\hat U\Psi\rangle = \langle\Psi|\hat U^\dagger \hat U|\Psi\rangle \Longrightarrow \boxed{\hat U^\dagger\hat U = \mathbb{1}}$ ... (1) unitary
Eigenstates of the Hamiltonian must be unchanged: $\hat H\Psi = \hat H\Psi' = \hat H\hat U\Psi = E\hat U\Psi = \hat U E\Psi = \hat U\hat H\Psi \Longrightarrow \boxed{[\hat H,\hat U]=0}$ ... (2)</td></tr>
<tr><td>Generator $\hat G$</td><td>Infinitesimal transformation: $\hat U = \mathbb 1 + i\varepsilon\hat G$ ... (3) $\mathbb 1 = \hat U^\dagger\hat U \overset{(3)}{\Rightarrow} \mathbb 1 = (\mathbb 1 - i\varepsilon\hat G^\dagger)(\mathbb 1 + i\varepsilon\hat G) = \mathbb 1 + i\varepsilon\hat G - i\varepsilon\hat G^\dagger + \cancel{\varepsilon^2\hat G^\dagger\hat G} \Longrightarrow$
$\mathbb 1 = \mathbb 1 + i\varepsilon(\hat G - \hat G^\dagger) \Longrightarrow \boxed{\hat G = \hat G^\dagger}$ ... (4) hermitian
(2) $\Longrightarrow \hat H\hat U - \hat U\hat H = 0 \overset{(3)}{\Rightarrow} \hat H(\mathbb 1 + i\varepsilon\hat G) - (\mathbb 1 + i\varepsilon\hat G)\hat H = 0 \Longrightarrow \cancel{\hat H} + i\varepsilon\hat H\hat G - \cancel{\hat H} - i\varepsilon\hat G\hat H \Longrightarrow \boxed{[\hat H,\hat G]=0}$ ... (5)</td></tr>
<tr><td>symmetry $\Longrightarrow$ conservation law</td><td>Schrödinger: $i\hbar\frac{\partial}{\partial t}\Psi = \hat H\Psi \overset{\hbar=1}{\Longrightarrow} i\frac{\partial}{\partial t}\Psi = \hat H\Psi \Longrightarrow \frac{\partial}{\partial t}\Psi = \frac1i\hat H\Psi \Longrightarrow \dot\Psi = -i\hat H\Psi$ ... (6a) $\Longrightarrow \dot\Psi^\dagger = i\Psi^\dagger\hat H$ ... (6b)
$\langle A\rangle = \int\Psi^\dagger\hat A\Psi\, d^3x \,\Big|\frac{\partial}{\partial t} \Longrightarrow \frac{\partial}{\partial t}\langle A\rangle = \frac{\partial}{\partial t}\int \Psi^\dagger\hat A\Psi\,d^3x \Longrightarrow \frac{\partial}{\partial t}\langle A\rangle = \int(\dot\Psi^\dagger\hat A\Psi + \Psi^\dagger\hat A\dot\Psi)\,d^3x \overset{(6ab)}{\Longrightarrow}$
$\frac{\partial}{\partial t}\langle A\rangle = \int(i\Psi^\dagger\hat H\hat A\Psi - i\Psi^\dagger\hat A\hat H\Psi)\,d^3x = i\int\Psi^\dagger(\hat H\hat A - \hat A\hat H)\Psi\,d^3x = i\langle\Psi|\hat H\hat A - \hat A\hat H|\Psi\rangle \Longrightarrow \boxed{\frac{\partial}{\partial t}\langle A\rangle = i\langle[\hat H,\hat A]\rangle}$ Ehrenfest
Now let $\hat A = \hat G$, then $\frac{\partial}{\partial t}\langle G\rangle = i\langle[\hat H,\hat G]\rangle \overset{(5)}{\Longrightarrow} \boxed{\frac{\partial}{\partial t}\langle G\rangle = 0}$</td></tr>
<tr><td>Example: translational invariance</td><td>1D: $x\to x+\varepsilon \Longrightarrow \Psi(x)\to\Psi'(x) = \Psi(x+\varepsilon) \overset{Taylor}{=} \Psi(x) + \varepsilon\frac{\partial\Psi}{\partial x} + \dots = \left(1+\varepsilon\frac{\partial}{\partial x}\right)\Psi(x) = \hat U\,\Psi(x) \overset{(3)}{=} (1+i\varepsilon\hat G)\,\Psi(x)$
$\Longrightarrow i\varepsilon\hat G \mathrel{\hat=} \varepsilon\frac{\partial}{\partial x} \Longrightarrow \boxed{\hat G \mathrel{\hat=} \frac1i\frac{\partial}{\partial x} = \hat p_x}$ ... translational invariance $\Longleftrightarrow$ conservation of momentum
Infinitesimal transformation: $\hat U_\varepsilon = \mathbb 1 + i\varepsilon\hat p_x$ Finite transformation: $\hat U(x_0) = \lim_{n\to\infty}\left(1 + i\frac{x_0}{n}\hat p_x\right)^n = e^{ix_0\hat p_x} = e^{x_0\frac{\partial}{\partial x}}$
$\Psi'(x) = \hat U\,\Psi(x) = e^{x_0\frac{\partial}{\partial x}}\Psi(x) \overset{Taylor}{=} \left(1 + x_0\frac{\partial}{\partial x} + \frac{x_0^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{x_0^3}{3!}\frac{\partial^3}{\partial x^3} + - \cdots\right)\Psi(x) \overset{Taylor}{=} \Psi(x+x_0)$</td></tr>
</table>

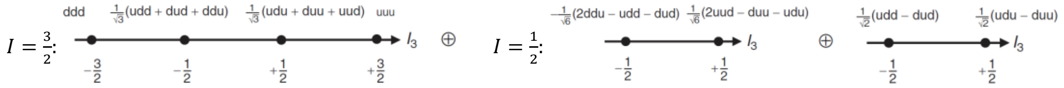
SU(2) Isospin Flavour Symmetry of the Strong Interaction (Isospin Representation of Quarks)

<table>
<tr><td>Idea:</td><td colspan="3">$\hat H = \hat H_0 + \hat H_{strong} + \hat H_{EM}$ because $\hat H_{EM} \ll \hat H_{strong}$, and $m_u \approx m_d$, there is an (ud) flavor symmetry.
We define $|u\rangle$ and $|d\rangle$ to be just two so-called Isospin-states of the same particle.</td></tr>
<tr><td>Analogy to normal spin</td><td colspan="3">“normal” spin-up-particle: $|\uparrow\rangle \mathrel{\hat=} \binom{1}{0} = \chi(s,m_s) = \chi\left(\frac12,+\frac12\right)$ $\Leftrightarrow$ up-quark isospin: $|u\rangle \mathrel{\hat=} \binom{1}{0} = \phi(I,I_3) = \phi\left(\frac12,+\frac12\right)$
“normal” spin-down-particle: $|\downarrow\rangle \mathrel{\hat=} \binom{0}{1} = \chi(s,m_s) = \chi\left(\frac12,-\frac12\right)$ $\Leftrightarrow$ down-quark isospin: $|d\rangle \mathrel{\hat=} \binom{0}{1} = \phi(I,I_3) = \phi\left(\frac12,-\frac12\right)$
“normal” $\hat{\vec S}^2$ operator: $\hat{\vec S}^2\,\chi(s,m_s) = s(s+1)\,\chi(s,m_s)$ $\Leftrightarrow$ isospin $\hat{\vec T}^2$ operator: $\hat{\vec T}^2\,\phi(I,I_3) = I(I+1)\,\phi(I,I_3)$
”normal“ $\hat S_z$ operator: $\hat S_z\,\chi(s,m_s) = m_s\,\chi(s,m_s)$ $\Leftrightarrow$ isospin $\hat T_3$ operator: $\hat T_3\,\phi(I,I_3) = I_3\,\phi(I,I_3)$</td></tr>
<tr><td>Symmetrie trafo SU(2)</td><td colspan="3">$|u\rangle \mathrel{\hat=} \binom{1}{0}^{\{T_3\}}$; $|d\rangle \mathrel{\hat=} \binom{0}{1}^{\{T_3\}}$ $\Longrightarrow \binom{u'}{d'} = \hat U(\vec\alpha)\binom{u}{d}$ with $\hat U(\vec\alpha) = e^{i\vec\alpha\cdot\vec T} \in SU(2) \Longrightarrow \hat U\hat U^\dagger = 1,\ \det(U) = 1$</td></tr>
<tr><td>Generators</td><td>$\hat U(\vec\alpha) = e^{i\alpha_i\hat G_i}$ with $\hat G_i \in \{\hat T_1,\hat T_2,\hat T_3\}$ and $\hat{\vec T} = \frac12\hat{\vec\sigma}$</td><td>Algebra:</td><td>$[\hat T_1,\hat T_2] = i\hat T_3$, $[\hat T_2,\hat T_3] = i\hat T_1$, $[\hat T_3,\hat T_1] = i\hat T_2 \Longrightarrow [\hat T_k,\hat T_l] = i\varepsilon_{klm}\hat T_m$</td></tr>
<tr><td>Ladder operators</td><td colspan="2">$\hat T_+|d\rangle = |u\rangle, \hat T_+|u\rangle = |0\rangle$ with $\hat T_+ = \hat T_1 + i\hat T_2$;
$\hat T_-|d\rangle = |0\rangle, \hat T_-|u\rangle = |d\rangle$ with $\hat T_- = \hat T_1 - i\hat T_2$</td><td></td></tr>
</table>

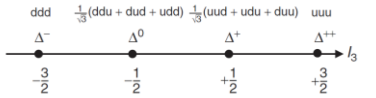
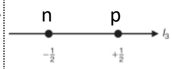
Combining 2 Up- or Down-Quarks or 2 Spin-Half Particles with SU(2)

<table>
<tr><td>Note</td><td colspan="5">*Note: Two-quark bound states combining only u and d are hypothetical, because the total color cannot add up to white.*
But this concept serves as a basis for combining three quarks to a Baryon state.</td></tr>
<tr><td>combining isospins</td><td>$I_3 = I_3^{(1)} + I_3^{(2)}$
$|I^{(1)} - I^{(1)}| \le I \le |I^{(1)} + I^{(1)}|$</td><td>combining spins</td><td>$m_s = m_s^{(1)} + m_s^{(2)}$
$|s^{(1)} - s^{(2)}| \le s \le |s^{(1)} + s^{(2)}|$</td><td>analogy</td><td>$M_J = m_s + m_l$
$|s - l| \le J \le |s + l|$</td></tr>
<tr><td>configuration space</td><td colspan="3">Both particles can each take on two different states ($|u\rangle, |d\rangle$)
Hence, the total configuration space is $\mathbf{2}\otimes\mathbf{2}$. Can be reduced to $\mathbf{3}_S\oplus\mathbf{1}_A$.</td><td colspan="2"></td></tr>
<tr><td>symmetric Isospin triplet</td><td colspan="5">• Maximum state $\phi_S(1,+1) = \phi^{(1)}\left(\frac12,+\frac12\right)\phi^{(2)}\left(\frac12,+\frac12\right) = |uu\rangle$ ... the S in ϕ_S stands for ”symmetric”
• $\phi_S(1,0) = \hat T_-\,\phi_S(1,+1) = \hat T_-|uu\rangle = \hat T_-^{(1)}|uu\rangle + \hat T_-^{(2)}|uu\rangle = |du\rangle + |ud\rangle \overset{norm.}{=} \frac{1}{\sqrt2}(|du\rangle + |ud\rangle)$.
• $\phi_S(1,-1) = \hat T_-(|du\rangle + |ud\rangle) = \hat T_-^{(1)}(|du\rangle + |ud\rangle) + \hat T_-^{(2)}(|du\rangle + |ud\rangle) = 0 + |dd\rangle + |dd\rangle + 0 \overset{norm.}{=} |dd\rangle$</td></tr>
<tr><td>Isospin singlet</td><td colspan="5">There is always just one state with $I_3 = I_3^{max}$, and just one with $I_3 = I_3^{min} = -I_3^{max}$. Hence, the 4th state can only be orthogonal to the “middle” state $\phi_S(1,0)$: $\phi_A(0,0) = \perp\,\phi_S(1,0) = \frac{1}{\sqrt2}(|du\rangle - |ud\rangle)$ ... the A in ϕ_A stands for ”antisymmetric”</td></tr>
<tr><td>Spin singlet and triplet</td><td colspan="5">Because the SU(2) algebra for combining spin-half is the same as for isospin, the possible spin wave functions of two quarks are constructed in the same manner. Hence, the combination of two spin-half particles gives:
spin triplet: $\chi_S(1,+1) = |\uparrow\uparrow\rangle, \chi_S(1,0) = \frac{1}{\sqrt2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \chi_S(1,-1) = |\downarrow\downarrow\rangle$; singlet: $\chi_A(0,0) = \frac{1}{\sqrt2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$</td></tr>
</table>

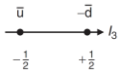
Combining 3 Up- or Down-Quarks or 3 Spin-Half Particles with SU(2)

Configuration Space	All three particles can each take on two different states ($\|u\rangle, \|d\rangle$). Hence, the total configuration space is $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$. This can be reduced to $\mathbf{4}_S\oplus\mathbf{2}_{MS}\oplus\mathbf{2}_{MA}$: A symmetric quadruplet, a mixed symmetric doublet, and a mixed antisymmetric doublet 
Symmetric Isospin Quadruplet	• Maximum state $\phi_S\left(\frac{3}{2},+\frac{3}{2}\right)=\|uuu\rangle$ • $\phi_S\left(\frac{3}{2},+\frac{1}{2}\right)=\hat{T}_-\,\phi_S\left(\frac{3}{2},+\frac{3}{2}\right)=\hat{T}_-\|uuu\rangle\overset{norm.}{=}\frac{1}{\sqrt{3}}(\|duu\rangle+\|udu\rangle+\|uud\rangle)$ • $\phi_S\left(\frac{3}{2},-\frac{1}{2}\right)=\hat{T}_-\,\phi_S\left(\frac{3}{2},+\frac{1}{2}\right)=\hat{T}_-(\|duu\rangle+\|udu\rangle+\|uud\rangle)\overset{norm.}{=}\frac{1}{\sqrt{3}}(\|ddu\rangle+\|dud\rangle+\|ddu\rangle)$ • $\phi_S\left(\frac{3}{2},-\frac{3}{2}\right)=\hat{T}_-\,\phi_S\left(\frac{3}{2},-\frac{1}{2}\right)=\hat{T}_-(\|udd\rangle+\|dud\rangle+\|ddu\rangle)\overset{norm.}{=}\|ddd\rangle$
Mixed Symmetric Doublet	There can be just one state with $I_3=I_3^{max}=\frac{3}{2}$, and just one with $I_3=I_3^{min}=-I_3^{max}=-\frac{3}{2}$. Therefore, we are looking for further two (mixed symmetric) states with $I=\frac{1}{2}$. • We take the symmetric two-quark state $\phi_S(1,+1)=\|uu\rangle$ which we couple with a third down-quark to reach I_3=$\frac{1}{2}$. • But we can also reach $I_3=\frac{1}{2}$ if we couple $\phi_S(1,0)=\frac{1}{\sqrt{2}}(\|du\rangle+\|ud\rangle)$ with a third up-quark. • One solution is a superposition of these states: $\phi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)=\alpha\|uu\rangle\|d\rangle+\beta(\|du\rangle+\|ud\rangle)\|u\rangle=\alpha\|uud\rangle+\beta(\|duu\rangle+\|udu\rangle)$. • Conditions to determine α and β: $\left\langle\phi_S\left(\frac{3}{2},+\frac{1}{2}\right)\middle\|\phi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)\right\rangle=0$ (orthogonality) and normalization $\alpha^2+\beta^2=1\Longrightarrow$ $\phi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)=\frac{1}{\sqrt{6}}(2\|uud\rangle-\|udu\rangle-\|duu\rangle)$ • We reach the second mixed symmetric state from here with $\hat{T}_-$: $\phi_{MS}\left(\frac{1}{2},-\frac{1}{2}\right)=\hat{T}_-\,\phi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)=\hat{T}_-(2\|uud\rangle-\|udu\rangle-\|duu\rangle)\overset{norm.}{=}\frac{1}{\sqrt{6}}(2\|ddu\rangle-\|dud\rangle-\|udd\rangle)$
Mixed Anti-Symmetric Doublet	There are further two (mixed antisymmetric) states with $I=\frac{1}{2}$. • We take the antisymmetric two-quark singlet state $\phi_A(1,0)$ which we couple with a third up-quark to reach I_3=$\frac{1}{2}$. $\phi_{MA}\left(\frac{1}{2},+\frac{1}{2}\right)=(\|du\rangle-\|ud\rangle)\|u\rangle=\frac{1}{\sqrt{2}}(\|udu\rangle-\|duu\rangle)$. • We reach the other mixed antisymmetric state with the $\hat{T}_-$ operator $\phi_{MA}\left(\frac{1}{2},-\frac{1}{2}\right)=\hat{T}_-\,\phi_{MA}\left(\frac{1}{2},+\frac{1}{2}\right)=\hat{T}_-(\|udu\rangle-\|duu\rangle)=\frac{1}{\sqrt{2}}(\|udd\rangle-\|dud\rangle)$.
Spin	Because the SU(2) algebra for combining spin-half is the same as for isospin, the possible spin wave functions of three quarks are constructed in the same manner. Hence, the combination of three spin-half particles give: - a symmetric Spin Quadruplet: $\chi_S\left(\frac{3}{2},+\frac{3}{2}\right)=\|\uparrow\uparrow\uparrow\rangle, \chi_S\left(\frac{3}{2},+\frac{1}{2}\right)=\frac{1}{\sqrt{3}}(\|\uparrow\uparrow\downarrow\rangle+\|\uparrow\downarrow\uparrow\rangle+\|\downarrow\uparrow\uparrow\rangle), \chi_S\left(\frac{3}{2},-\frac{1}{2}\right)=\frac{1}{\sqrt{3}}(\|\uparrow\downarrow\downarrow\rangle+\|\downarrow\uparrow\downarrow\rangle+\|\downarrow\downarrow\uparrow\rangle), \chi_S\left(\frac{3}{2},-\frac{3}{2}\right)=\|\downarrow\downarrow\downarrow\rangle$ - a mixed symmetric Spin Doublet: $\chi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)=\frac{1}{\sqrt{6}}(2\|\uparrow\uparrow\downarrow\rangle-\|\uparrow\downarrow\uparrow\rangle-\|\downarrow\uparrow\uparrow\rangle), \chi_{MS}\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{\sqrt{6}}(2\|\downarrow\downarrow\uparrow\rangle-\|\downarrow\uparrow\downarrow\rangle-\|\uparrow\downarrow\downarrow\rangle)$ - a mixed antisymmetric Spin Doublet: $\chi_{MA}\left(\frac{1}{2},+\frac{1}{2}\right)=\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\uparrow\rangle-\|\downarrow\uparrow\uparrow\rangle), \chi_{MA}\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\downarrow\rangle-\|\downarrow\uparrow\downarrow\rangle)$

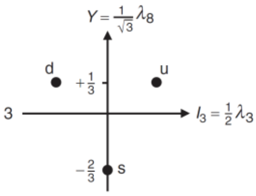
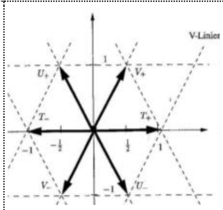
Ground State Baryon Wave Functions

Symmetries	$\Psi = \phi_{flavor}\chi_{spin}\xi_{color}\eta_{space}$ Quarks are fermions $\Longrightarrow$ $\Psi \overset{!}{=} \Psi_A$. Always: $\xi_{color} = \xi_A$ with $L = 0$: $\eta_{space} = \eta_S \Longrightarrow$ $\Psi_A = \phi_{flavor}\chi_{spin}\xi_A\eta_S \Longrightarrow \phi_{flavor}\chi_{spin}$ must be symmetric (for $L = 0$)	
Δ-Baryons	$\Psi_A = \phi_S\chi_S\xi_A\eta_S$ $I = \frac{3}{2}, s = \frac{3}{2}$	
Neutron Proton	$\Psi_A = \frac{1}{\sqrt{2}}\underbrace{(\phi_{MS}\chi_{MS} + \phi_{MA}\chi_{MA})}_{symmetric}\xi_A\eta_S$ $I = \frac{1}{2}, s = \frac{1}{2}$	
spin-up Proton	$\lvert p\uparrow\rangle = \frac{1}{\sqrt{2}}\left(\phi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right)\chi_{MS}\left(\frac{1}{2},+\frac{1}{2}\right) + \phi_{MA}\left(\frac{1}{2},+\frac{1}{2}\right)\chi_{MA}\left(\frac{1}{2},+\frac{1}{2}\right)\right) \Longrightarrow$ $\lvert p\uparrow\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}(2\lvert uud\rangle - \lvert udu\rangle - \lvert duu\rangle)\frac{1}{\sqrt{6}}(2\lvert\uparrow\uparrow\downarrow\rangle - \lvert\uparrow\downarrow\uparrow\rangle - \lvert\downarrow\uparrow\uparrow\rangle) + \frac{1}{\sqrt{2}}(\lvert udd\rangle - \lvert dud\rangle)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\uparrow\rangle - \lvert\downarrow\uparrow\uparrow\rangle)\right) \Longrightarrow$ $\lvert p\uparrow\rangle = \frac{1}{\sqrt{18}}(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + 2d\downarrow u\uparrow u\uparrow - d\uparrow u\uparrow u - d\uparrow u\downarrow u\uparrow)$	

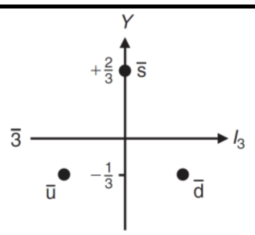
Isospin Representation of Anti-Quarks

Definition:	$\lvert\bar{u}\rangle \mathrel{\hat{=}} \begin{pmatrix}0\\1\end{pmatrix}; \lvert\bar{d}\rangle \mathrel{\hat{=}} \begin{pmatrix}-1\\0\end{pmatrix} \Longrightarrow \lvert\bar{q}\rangle \mathrel{\hat{=}} \begin{pmatrix}-\bar{d}\\\bar{u}\end{pmatrix}$	Symmetry Trafo: $\lvert\bar{q}'\rangle = \hat{U}(\vec{\alpha})\,\lvert\bar{q}'\rangle$ with $\hat{U}(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}}$ 
Ladder oper.	$\hat{T}_+\lvert\bar{u}\rangle = -\lvert\bar{d}\rangle, \hat{T}_+\lvert\bar{d}\rangle = \lvert 0\rangle$ with $\hat{T}_+ = \hat{T}_1 + i\hat{T}_2$; $\hat{T}_-\lvert\bar{u}\rangle = \lvert 0\rangle, \hat{T}_-\lvert\bar{d}\rangle = \lvert\bar{u}\rangle$ with $\hat{T}_- = \hat{T}_1 - i\hat{T}_2$	

SU(3) Flavor Symmetry of the Strong Interaction

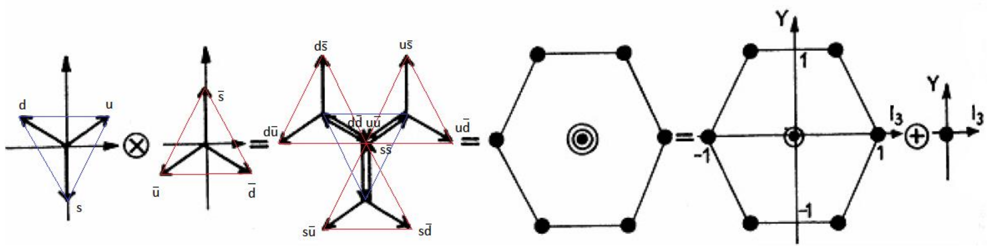
Idea:	The SU(2) flavor symmetry is almost exact, as $m_u \approx m_d$. As $\hat{H}_{strong}$ treats uds equally, we can include the strange-quark with SU(3). However, the symmetry is not perfect, because the difference between m_s and $m_{u/d}$ is approx. 100 MeV
Symmetry Trafo	$\begin{pmatrix}u'\\d'\\s'\end{pmatrix} = \hat{U}\begin{pmatrix}u\\d\\s\end{pmatrix}$ with $\hat{U}$ ... 3x3 matrix; $\hat{U}\hat{U}^\dagger = \mathbb{1}$, $\det(\hat{U}) = 1$ | $\hat{U} = e^{i\vec{\alpha}\cdot\vec{T}} = e^{i\alpha_i\hat{T}_i}$; $i = 1 \ldots 8$ with $\boxed{\hat{T}_i = \frac{1}{2}\hat{\lambda}_i}$
Generators (Gell-Mann matrices)	SU(3) requires $3^2 - 1 = 8$ generators. (Remark: A complex 3x3-matrix would have 18 parameters. The condition $\boxed{\hat{U}\hat{U}^\dagger = \mathbb{1}}$ reduces this to 9 parameters; $\boxed{\det(\hat{U}) = 1}$ reduces further to 8 parameters). Suitable generators are e.g. the Gell-Mann matrices $\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\lambda}_5, \hat{\lambda}_6, \hat{\lambda}_7, \hat{\lambda}_8\}$ with $\boxed{\mathrm{Tr}(\hat{\lambda}_i) = 0}$. Further: $\boxed{\mathrm{Tr}(\hat{\lambda}_i\hat{\lambda}_j) = 2\delta_{ij}}$ $\hat{\lambda}_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \mathrel{\hat{=}} \hat{\sigma}_1$; $\hat{\lambda}_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \mathrel{\hat{=}} \hat{\sigma}_2$; $\hat{\lambda}_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \mathrel{\hat{=}} \hat{\sigma}_3 \Longrightarrow$ Isospin $\boxed{\hat{T}_3 = \frac{1}{2}\hat{\lambda}_3}$ $\hat{\lambda}_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$; $\hat{\lambda}_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}$ $\hat{\lambda}_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$; $\hat{\lambda}_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$; $\hat{\lambda}_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix} \Longrightarrow$ Hyper Charge $\boxed{\hat{Y} = \frac{1}{\sqrt{3}}\hat{\lambda}_8}$
Isospin and Hypercharge	$[\hat{T}_3, \hat{Y}] = 0 \Longrightarrow$ Multipletts can be described with I and Y: $\hat{T}_3\lvert I_3, Y\rangle = I_3\lvert I_3, Y\rangle$ and $\hat{Y}\lvert I_3, Y\rangle = Y\lvert I_3, Y\rangle$  $\hat{T}_3\lvert u\rangle = +\frac{1}{2}\lvert u\rangle$ $\hat{Y}\lvert u\rangle = +\frac{1}{3}\lvert u\rangle$ $\hat{T}_3\lvert d\rangle = -\frac{1}{2}\lvert d\rangle$ $\hat{Y}\lvert d\rangle = +\frac{1}{3}\lvert d\rangle$ $\hat{T}_3\lvert s\rangle = 0$ $\hat{Y}\lvert s\rangle = -\frac{2}{3}\lvert s\rangle$ Total Isospin: $\hat{T}^2 = \sum_{i=1}^8 \hat{T}_i^2 = \frac{1}{4}\sum_{i=1}^8 \hat{\lambda}_i^2 = \frac{4}{3}\mathbb{1}_3$ Ladder Operators $\hat{T}_\pm = \hat{T}_1 \pm i\hat{T}_2$ $\hat{V}_\pm = \hat{T}_4 \pm i\hat{T}_5$ $\hat{U}_\pm = \hat{T}_6 \pm i\hat{T}_7$  $\hat{V}_+\lvert s\rangle = \lvert u\rangle$ $\hat{V}_-\lvert u\rangle = \lvert s\rangle$ $\hat{U}_+\lvert s\rangle = \lvert d\rangle$ $\hat{U}_-\lvert d\rangle = \lvert s\rangle$ $\hat{T}_+\lvert d\rangle = \lvert u\rangle$ $\hat{T}_-\lvert u\rangle = \lvert d\rangle$

Antiquarks

Idea:	 $\hat{T}_3\lvert\bar{u}\rangle = -\frac{1}{2}\lvert\bar{u}\rangle$ $\hat{\bar{V}}_+\lvert\bar{u}\rangle = -\lvert\bar{s}\rangle$ $\hat{T}_3\lvert\bar{d}\rangle = +\frac{1}{2}\lvert\bar{d}\rangle$ $\hat{\bar{V}}_-\lvert\bar{s}\rangle = -\lvert\bar{u}\rangle$ $\hat{T}_3\lvert\bar{s}\rangle = 0$ $\hat{\bar{U}}_+\lvert\bar{d}\rangle = -\lvert\bar{s}\rangle$ $\hat{Y}\lvert\bar{u}\rangle = -\frac{1}{3}\lvert\bar{u}\rangle$ $\hat{\bar{U}}_-\lvert\bar{s}\rangle = -\lvert\bar{d}\rangle$ $\hat{Y}\lvert\bar{d}\rangle = -\frac{1}{3}\lvert\bar{d}\rangle$ $\hat{\bar{T}}_+\lvert\bar{u}\rangle = -\lvert\bar{d}\rangle$ $\hat{Y}\lvert\bar{s}\rangle = +\frac{2}{3}\lvert\bar{s}\rangle$ $\hat{\bar{T}}_-\lvert\bar{d}\rangle = -\lvert\bar{u}\rangle$ The elementary antiquark-triplet can be identified with the antiquark states $\lvert\bar{u}\rangle, \lvert\bar{d}\rangle$ and $\lvert\bar{s}\rangle$. They cannot be expressed in the quark basis $\{\lvert u\rangle, \lvert d\rangle, \lvert s\rangle\}$ The antiquark basis $\lvert\bar{u}\rangle \to \begin{pmatrix}1\\0\\0\end{pmatrix}$; $\lvert\bar{d}\rangle \to \begin{pmatrix}0\\1\\0\end{pmatrix}$; $\lvert\bar{s}\rangle \to \begin{pmatrix}0\\0\\1\end{pmatrix}$ has different Generators $\{\hat{\bar{T}}_1, \hat{\bar{T}}_2, \hat{\bar{T}}_3, \hat{\bar{T}}_4, \hat{\bar{T}}_5, \hat{\bar{T}}_6, \hat{\bar{T}}_7, \hat{\bar{T}}_8\}$ with $\boxed{\hat{\bar{T}}_i = -\hat{T}_i^*}$

The Light Mesons: coupling quark and antiquark to a $q\bar{q}$ octet (and a $q\bar{q}$ singlet)

- By coupling a quark triplet $\mathbf{3}$ with an antiquark triplet $\overline{\mathbf{3}}$, altogether $\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{9}$ quark-antiquark states can be produced, which represent mesons (bound quark states with even number of quarks: here with one quark and one antoquark). These 9 states split into an octet and a singlet: $\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, as is derived below.
- Graphical derivation: The vertices of the quark triplet $\mathbf{3}$ are the center-points for the three coupled anti-triplets $\overline{\mathbf{3}}$. This creates a hexagon with six uniquely occupied corners and a triple-occupied center.



- In a SU (3) multiplet, each point of the outer shell is occupied once, and each point of the next inner shell (if the outer shell was not a triangle) is occupied twice. The outer shell here is a hexagon; the next inner shell is the center, which should therefore only be occupied twice (instead of three-fold). Therefore, the hexagon with the triple-occupied center can be decomposed into a hexagon with a double-occupied center (an octet) and a singlet (the additionally occupied center point).
- The six border states $\phi_8(I, I_3, Y)$ of the octet are unique (quantum number $I = I_3^{max} = 1$):

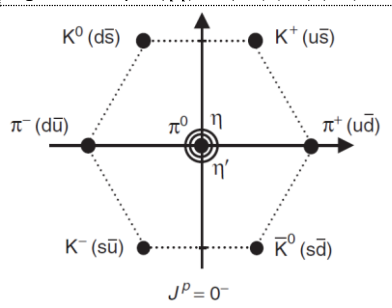
$$\boxed{\phi_8\left(1,-\tfrac{1}{2},+1\right) = |d\bar{s}\rangle, \phi_8\left(1,+\tfrac{1}{2},+1\right) = |u\bar{s}\rangle, \phi_8(1,+1,0) = |u\bar{d}\rangle, \phi_8\left(1,+\tfrac{1}{2},-1\right) = |s\bar{d}\rangle, \phi_8\left(1,-\tfrac{1}{2},-1\right) = |s\bar{u}\rangle, \phi_8\left(1,-\tfrac{1}{2},0\right) = |d\bar{u}\rangle}$$

- The first center state $\phi_8^{C1}(1,0,0)$ can be derived by using ladder operators.
  We choose $\phi_8^{C1}(0,1,0) = \hat{T}_-|u\bar{d}\rangle = \hat{T}_-^{(1)}|u\bar{d}\rangle + \hat{T}_-^{(2)}|u\bar{d}\rangle \overset{norm}{\Longrightarrow} \boxed{\phi_8^{C1}(1,0,0) = \frac{1}{\sqrt{2}}\left(|d\bar{d}\rangle - |u\bar{u}\rangle\right)}$ ... (1)
- The second center state $\phi_8^{C2}(1,0,0)$ can be derived by means of further ladder operators:
  $\hat{V}_-|u\bar{s}\rangle \overset{norm}{=} \frac{1}{\sqrt{2}}(|s\bar{s}\rangle - |u\bar{u}\rangle)$ ... (2) $\hat{U}_-|d\bar{s}\rangle \overset{norm}{=} \frac{1}{\sqrt{2}}\left(|s\bar{s}\rangle - |d\bar{d}\rangle\right)$ ... (3) Superposition: $\phi_8^{C2}(0,0,1) = \alpha\frac{|s\bar{s}\rangle - |u\bar{u}\rangle}{\sqrt{2}} + \beta\frac{|s\bar{s}\rangle - |d\bar{d}\rangle}{\sqrt{2}}$ ... (4)
  with : $\langle C1|C2\rangle = 0$ and $\alpha^2 + \beta^2 = 1 \Longrightarrow \boxed{\phi_8^{C2}(1,0,0) = \frac{1}{\sqrt{6}}\left(2|s\bar{s}\rangle - |u\bar{u}\rangle - |d\bar{d}\rangle\right)}$

The Light Meson $q\bar{q}$ singlet

- Remaining center state is a singlet. We take the ansatz: $|\mathbf{1}\rangle = \alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle$ ... (1)
  We know: Any ladder operator acting on $|\mathbf{1}\rangle$ must result in 0. Therefore: $\hat{T}_+|\mathbf{1}\rangle = 0 \overset{(1)}{\Rightarrow} \hat{T}_+\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow$
  $\left(\hat{T}_+^{(1)} + \hat{T}_+^{(2)}\right)\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow \hat{T}_+^{(1)}\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) + \hat{T}_+^{(2)}\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow$
  $\left(0 + \beta|u\bar{d}\rangle + 0\right) + \left(-\alpha|u\bar{d}\rangle + 0 + 0\right) = 0 \Longrightarrow \beta|u\bar{d}\rangle - \alpha|u\bar{d}\rangle = 0 \Longrightarrow \beta|u\bar{d}\rangle = \alpha|u\bar{d}\rangle \Longrightarrow \alpha = \beta$ ... (2)
  Again: Any ladder operator acting on $|\mathbf{1}\rangle$ must result in 0. Therefore: $\hat{V}_-|\mathbf{1}\rangle = 0 \overset{(1)}{\Rightarrow} \hat{V}_-\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow$
  $\left(\hat{V}_-^{(1)} + \hat{V}_-^{(2)}\right)\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow \hat{V}_-^{(1)}\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) + \hat{V}_-^{(2)}\left(\alpha|u\bar{u}\rangle + \beta|d\bar{d}\rangle + \gamma|s\bar{s}\rangle\right) = 0 \Longrightarrow$
  $\left(\alpha|s\bar{u}\rangle + 0 + 0\right) + \left(0 + 0 - \gamma|s\bar{u}\rangle\right) = 0 \Longrightarrow \alpha|s\bar{u}\rangle - \gamma|s\bar{u}\rangle = 0 \Longrightarrow \alpha = \gamma$ ... (3) (1) $\overset{(2),(3)}{\Longrightarrow} |\mathbf{1}\rangle = \alpha|u\bar{u}\rangle + \alpha|d\bar{d}\rangle + \alpha|s\bar{s}\rangle$ ... (4)
  Normalization: $\langle\mathbf{1}|\mathbf{1}\rangle = 1 \overset{(4)}{\Rightarrow} \alpha^2 + \alpha^2 + \alpha^2 = 3\alpha^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{3}} \overset{(4)}{\Rightarrow} \boxed{|\mathbf{1}\rangle = \phi_1(0,0,0) = \frac{1}{\sqrt{3}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle\right)}$ (as $I = I_3^{max} = 0$)

L=0 Mesons: Pseudoscalar Mesons

	Quark/antiquark states with orbital angular momentum $\boxed{\ell = 0}$, and spin $\boxed{s = 0}$ $\Longrightarrow$total angular momentum $\boxed{J = \ell + s = 0}$ $\Longrightarrow$ Combining anti-symmetrized states ϕ_8 and ϕ_1 with anti-symmetric singlet spin state $\chi_A(0,0) = \frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ Negative Parity: $P(q\bar{q}) = (+1)(-1)(-1)^\ell = -1$, Mass differences because $m_s > m_{u/d}$
Pseudo-scalar Mesons	 π^0: $135\ MeV$ ... $\frac{1}{\sqrt{4}}\left(\lvert u\bar{u}\rangle - \lvert d\bar{d}\rangle - \lvert\bar{u}u\rangle + \lvert\bar{d}d\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ π^-: $140\ MeV$ ... $\frac{1}{\sqrt{2}}(\lvert d\bar{u}\rangle - \lvert\bar{u}d\rangle)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ π^+: $140\ MeV$ ... $\frac{1}{\sqrt{2}}\left(\lvert u\bar{d}\rangle - \lvert\bar{d}u\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ K^-: $494\ MeV$ ... $\frac{1}{\sqrt{2}}(\lvert s\bar{u}\rangle - \lvert\bar{u}s\rangle)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ K^+: $494\ MeV$ ... $\frac{1}{\sqrt{2}}\left(\lvert s\bar{d}\rangle - \lvert\bar{d}s\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ K^0: $498\ MeV$ ... $\frac{1}{\sqrt{2}}(\lvert d\bar{s}\rangle - \lvert\bar{s}d\rangle)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ $\overline{K}^0$: $498\ MeV$ ... $\frac{1}{\sqrt{2}}\left(\lvert s\bar{d}\rangle - \lvert\bar{d}s\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ η : $548\ MeV$ ... $\frac{1}{\sqrt{12}}\left(\lvert u\bar{u}\rangle + \lvert d\bar{d}\rangle - 2\lvert s\bar{s}\rangle - \lvert\bar{u}u\rangle - \lvert\bar{d}d\rangle + 2\lvert\bar{s}s\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$ η' : $958\ MeV$ ... $\frac{1}{\sqrt{12}}\left(\lvert u\bar{u}\rangle + \lvert d\bar{d}\rangle + \lvert s\bar{s}\rangle - \lvert\bar{u}u\rangle - \lvert\bar{d}d\rangle - \lvert\bar{s}s\rangle - \lvert\bar{u}u\rangle - \lvert\bar{d}d\rangle - \lvert\bar{s}s\rangle + \lvert u\bar{u}\rangle + \lvert d\bar{d}\rangle + \lvert s\bar{s}\rangle\right)\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$
Masses:	$m = m_1 + m_2 + \frac{A}{m_1 m_2}\langle\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2\rangle = m_1 + m_2 + \frac{A}{m_1 m_2}\frac{1}{2}\left(\langle\hat{\vec{S}}^2\rangle - \langle\hat{\vec{S}}_1^2\rangle - \langle\hat{\vec{S}}_2^2\rangle\right) = m_1 + m_2 + \frac{A}{m_1 m_2}\frac{1}{2}\left(0 - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right) \Longrightarrow$ $\boxed{m = m_1 + m_2 - \frac{3}{4}\frac{A}{m_1 m_2}}$ $m_d = m_u = 0.307GeV, m_s = 0.490\ GeV, A = 0.06GeV^3$ (does not explain the η' mass properly!)

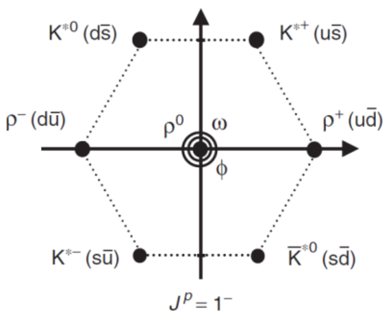
L=0 Mesons: Vector Mesons

Vector Mesons

Quark/antiquark states with orbital angular momentum $\boxed{\ell = 0}$, and spin $\boxed{s = 1}$ $\Longrightarrow$ total angular momentum $\boxed{J = \ell + s = 1}$ $\Longrightarrow$

Combining symmetrized ϕ_8 and ϕ_1 states with one of the symmetric triplet spin states $\chi_S(1,+1)$, $\chi_S(1,0)$, or $\chi_S(1,-1)$

Negative Parity: $P(q\bar{q}) = (+1)(-1)(-1)^{\ell} = -1$. Higher masses because of spin-spin interactions



$\rho^0\ :775\ MeV \ldots \frac{1}{\sqrt{4}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle + |\bar{u}u\rangle - |\bar{d}d\rangle\right)\chi_S$

$\rho^-\ :775\ MeV \ldots \frac{1}{\sqrt{2}}\left(|d\bar{u}\rangle + |\bar{u}d\rangle\right)\chi_S$

$\rho^+\ :775\ MeV \ldots \frac{1}{\sqrt{2}}\left(|u\bar{d}\rangle + |\bar{d}u\rangle\right)\chi_S$

$K^{*-}: 892\ MeV \ldots \frac{1}{\sqrt{2}}\left(|s\bar{u}\rangle + |\bar{u}s\rangle\right)\chi_S$

$K^{*+}: 892\ MeV \ldots \frac{1}{\sqrt{2}}\left(|s\bar{d}\rangle + |\bar{d}s\rangle\right)\chi_S$

$K^{*0}: 896\ MeV \ldots \frac{1}{\sqrt{2}}\left(|d\bar{s}\rangle + |\bar{s}d\rangle\right)\chi_S$

$\bar{K}^{*0}: 896\ MeV \ldots \frac{1}{\sqrt{2}}\left(|s\bar{d}\rangle + |\bar{d}s\rangle\right)\chi_S$

$\omega\ :783\ MeV \ldots \frac{1}{\sqrt{2}}|u\bar{u} + d\bar{d}\rangle\chi_S$ (sic!)

$\Phi : 1020\ MeV \ldots |s\bar{s}\rangle\chi_S$ (sic!)

observed ω and Φ are mixtures of ϕ_8 and ϕ_1 states!

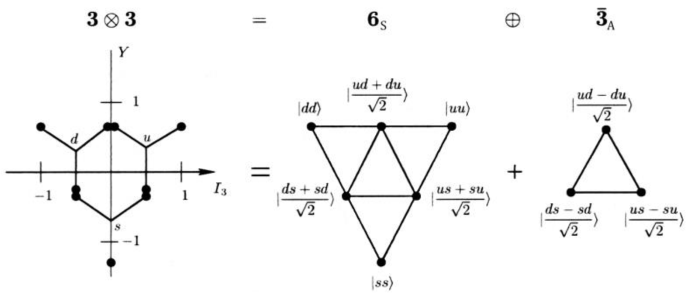
Masses:

$$m = m_1 + m_2 + \frac{A}{m_1 m_2}\frac{1}{2}\left(\langle\hat{S}^2\rangle - \langle\hat{S}_1^2\rangle - \langle\hat{S}_2^2\rangle\right) = m_1 + m_2 + \frac{A}{m_1 m_2}\frac{1}{2}\left(1(1+1) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right) \Longrightarrow$$

$\boxed{m = m_1 + m_2 + \frac{1}{4}\frac{A}{m_1 m_2}}$ $m_d = m_u = 0.307GeV, m_s = 0.490\ GeV, A = 0.06GeV^3$

Coupling two quarks to a (hypothetical) qq sextet and qq anti-triplet

Note: Two-quark bound states combining u, d and s are hypothetical, because the total color cannot add up to white.
But this concept serves as a basis for combining three quarks to a triplet Baryon state.



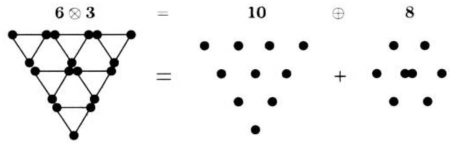
- By coupling a quark-triplet **3** with another quark-triplet **3**, altogether **9** quark states can be formed. They split up into a sextet and an anti-triplet: $\mathbf{3}\otimes\mathbf{3} = \mathbf{6_S} \oplus \bar{\mathbf{3}}_A$, as derived here.
- Graphical derivation: The corners of the first quark triplet **3** are the center points of the second coupled triplet **3**. Thereby, a triangular-shaped sextet $\mathbf{6_S}$ with six uniquely occupied outer border states and yet another triangular anti-triplet $\bar{\mathbf{3}}_A$ with three uniquely occupied states emerge.
- The three corners of the sextet $\mathbf{6_S}$ $\phi_{6_S}(I, I_3, Y)$ (quantum number $I = I_3^{max} = 1$) can be read off directly: $\boxed{\phi_{6_S}\left(1,+1,+\frac{2}{3}\right) = |uu\rangle, \phi_{6_S}\left(1,-1,+\frac{2}{3}\right) = |dd\rangle, \phi_{6_S}\left(1,0,-\frac{4}{3}\right) = |ss\rangle}$
- The other three sextet $\mathbf{6_S}$ states can be reached by means of ladder operators:
  - $\phi_{6_S}\left(1,0,+\frac{2}{3}\right) = \hat{T}_+|dd\rangle = \hat{T}_+^{(1)}|dd\rangle + \hat{T}_+^{(2)}|dd\rangle \overset{norm}{\Longrightarrow} \boxed{\phi_{6_S}\left(1,0,+\frac{2}{3}\right) = \frac{1}{\sqrt{2}}\left(|ud\rangle + |du\rangle\right)}$
  - $\phi_{6_S}\left(1,-\frac{1}{2},-\frac{1}{3}\right) = \hat{U}_-|dd\rangle = \hat{U}_-^{(1)}|dd\rangle + \hat{U}_-^{(2)}|dd\rangle \overset{norm}{\Longrightarrow} \boxed{\phi_{6_S}\left(1,-\frac{1}{2},-\frac{1}{3}\right) = \frac{1}{\sqrt{2}}\left(|sd\rangle + |ds\rangle\right)}$
  - $\phi_{6_S}\left(1,+\frac{1}{2},-\frac{1}{3}\right) = \hat{V}_-|uu\rangle = \hat{V}_-^{(1)}|uu\rangle + \hat{V}_-^{(2)}|uu\rangle \overset{norm}{\Longrightarrow} \boxed{\phi_{6_S}\left(1,+\frac{1}{2},-\frac{1}{3}\right) = \frac{1}{\sqrt{2}}\left(|su\rangle + |us\rangle\right)}$
- The first anti-triplet state results from normalization and orthogonality condition:

  Ansatz: $\phi_{\bar{3}_A}\left(0,0,+\frac{2}{3}\right) = \alpha|ud\rangle + \beta|du\rangle \ldots (1)$ orthogonality: $\left\langle\phi_{6_S}\left(1,0,+\frac{2}{3}\right)\middle|\phi_{\bar{3}_A}\left(0,0,+\frac{2}{3}\right)\right\rangle = 0 \overset{(1)}{\Rightarrow} \frac{\langle ud| + \langle du|}{\sqrt{2}}\left(\alpha|ud\rangle + \beta|du\rangle\right) = 0 \Longrightarrow$

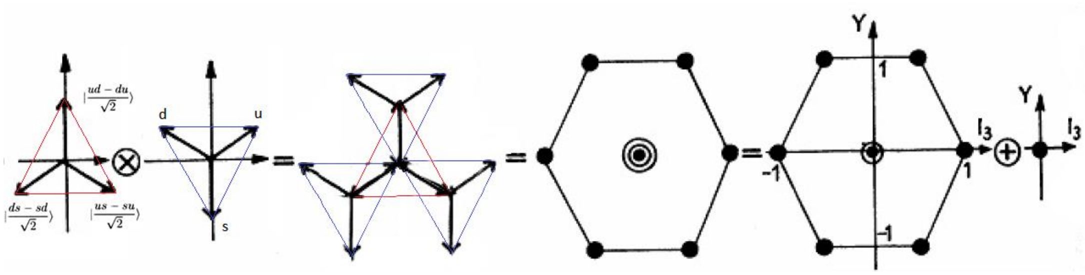
  $\frac{\alpha}{\sqrt{2}}\langle ud|ud\rangle + \frac{\beta}{\sqrt{2}}\langle du|du\rangle = 0 \Longrightarrow \beta = -\alpha \overset{(1)}{\Rightarrow} \phi_{\bar{3}_A}\left(0,0,+\frac{2}{3}\right) = \alpha|ud\rangle - \alpha|du\rangle \ldots (2)$

  Normalization: $\alpha^2 + \alpha^2 = 2\alpha^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{2}} \overset{(2)}{\Rightarrow} \boxed{\phi_{\bar{3}_A}\left(+\frac{2}{3},0,0\right) = \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right)}$
- The other two anti-triplet states can be reached by means of ladder operators:
  - $\phi_{\bar{3}_A}\left(0,-\frac{1}{2}-\frac{1}{3}\right) = \hat{V}_-\frac{|ud\rangle-|du\rangle}{\sqrt{2}} = \hat{V}_-^{(1)}\frac{|ud\rangle-|du\rangle}{\sqrt{2}} + \hat{V}_-^{(2)}\frac{|ud\rangle-|du\rangle}{\sqrt{2}} = \frac{|sd\rangle-0}{\sqrt{2}} + \frac{0-|ds\rangle}{\sqrt{2}} \overset{norm}{\Longrightarrow} \boxed{\phi_{\bar{3}_A}\left(0,-\frac{1}{2},-\frac{1}{3},\right) = \frac{1}{\sqrt{2}}\left(|sd\rangle - |ds\rangle\right)}$
  - $\phi_{\bar{3}_A}\left(0,+\frac{1}{2},-\frac{1}{3}\right) = \hat{U}_-\frac{|ud\rangle-|du\rangle}{\sqrt{2}} = \hat{U}_-^{(1)}\frac{|ud\rangle-|du\rangle}{\sqrt{2}} + \hat{U}_-^{(2)}\frac{|ud\rangle-|du\rangle}{\sqrt{2}} = \frac{0-|su\rangle}{\sqrt{2}} + \frac{|us\rangle-0}{\sqrt{2}} \overset{norm}{\Longrightarrow} \boxed{\phi_{\bar{3}_A}\left(0,+\frac{1}{2},-\frac{1}{3}\right) = \frac{1}{\sqrt{2}}\left(|us\rangle - |su\rangle\right)}$

Coupling 3 quarks to L=0 qqq Baryons



- We are looking for the reduction of $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ (these are 27 states). We have just derived: $\mathbf{3}\otimes\mathbf{3} = \mathbf{6}_S \oplus \overline{\mathbf{3}}_A$, therefore: $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}_S \oplus \overline{\mathbf{3}}_A)\otimes\mathbf{3} \Longrightarrow$ $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{6}_S\otimes\mathbf{3} \oplus \overline{\mathbf{3}}_A\otimes\mathbf{3}$
- Coupling $\mathbf{6}_S\otimes\mathbf{3}$ (18 states, see graphics on the right side): The points of the symmetric quark sextet $\mathbf{6}_S$ are the center points of the quark triplet $\mathbf{3}$ that we are coupling with the sextet. By this, a triangle with three uniquely occupied corners, six double-occupied points alongside the outer border, and triple occupied center point emerges. However, with a triangular multiplet, all points on the outer border can only be occupied once. Therefore, the described triangle falls apart into a triangular, <u>symmetric decuplet</u> $\mathbf{10}_S$ (with nine uniquely occupied points at the border and one uniquely occupied center point) and a hexagonal, <u>mixed-symmetric octet</u> $\mathbf{8}_{MS}$ (with six uniquely occupied corner points, and a doubly occupied center point). Hence: $\boxed{\mathbf{6}_S\otimes\mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{MS}}$
- Coupling $\overline{\mathbf{3}}_A\otimes\mathbf{3}$ (9 states): The points of the antisymmetric triplet $\overline{\mathbf{3}}_A$ are the center points of the quark triplet $\mathbf{3}$ that we are coupling with the antisymmetric triplet (see below).

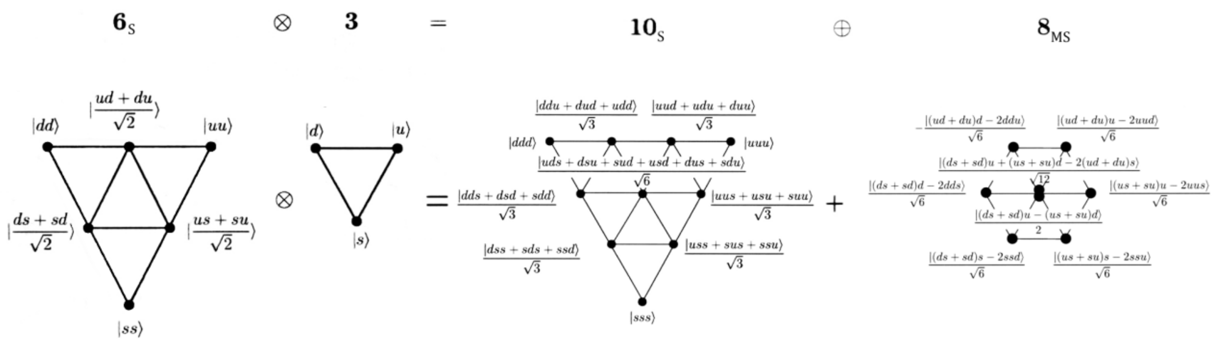


- In a SU(3) multiplet, every point on the outer border is occupied only once, and every point on the next inner layer (if the outer border is not a triangle) is occupied twice. Here, the outer border has a hexagonal shape, and the next inner layer is already the center point, which therefore only can be occupied twice instead of three times. Therefore, the hexagon with the triple occupied center point falls apart into a hexagon-shaped <u>mixed-symmetric octet</u> $\mathbf{8}_{MA}$ with a doubly occupied center point, and a <u>totally antisymmetric singlet</u> $\mathbf{1}_A$.
  Hence: $\boxed{\overline{\mathbf{3}}_A\otimes\mathbf{3} = \mathbf{8}_{MA} \oplus \mathbf{1}_A}$
- In total, the three-quark-multiplet with 27 states can be decomposed as follows:
  $\boxed{\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}_S \oplus \overline{\mathbf{3}}_A)\otimes\mathbf{3} = \mathbf{6}_S\otimes\mathbf{3} \oplus \overline{\mathbf{3}}_A\otimes\mathbf{3} = (\mathbf{10}_S \oplus \mathbf{8}_{MS}) \oplus (\mathbf{8}_{MA} \oplus \mathbf{1}_A)}$

Symmetry	$\Psi = \phi_{flavor}\chi_{spin}\xi_{color}\eta_{space}$ Quarks are fermions $\Longrightarrow$ $\Psi \overset{!}{=} \Psi_A$. Always: $\xi_{color} = \xi_A$ with $L = 0$: $\eta_{space} = \eta_S \Longrightarrow$ $\Psi_A = \phi_{flavor}\chi_{spin}\xi_A\eta_S \Longrightarrow \phi_{flavor}\chi_{spin}$ must be symmetric (for $L = 0$) $\Longrightarrow$ singlet $\mathbf{1}_A$ cannot exist in ground state $L = 0$, because there is no anti-symmetric 3 particle spin function χ_A

The Symmetric Decuplet and Mixed Symmetric Octet of Light Baryon States

- As described above: By coupling $\mathbf{6_S}\otimes\mathbf{3}$ (18 states) a triangular (symmetric) decuplet $\mathbf{10_S}$ and a hexagonal (mixed symmetric) octet $\mathbf{8_{MS}}$ emerges:



- The three corner-points of the decuplet $\mathbf{10_S}$ can be read off directly $\boxed{|\mathbf{10}_1\rangle = |ddd\rangle, |\mathbf{10}_2\rangle = |uuu\rangle, |\mathbf{10}_3\rangle = |sss\rangle}$
- The other six border points of the decuplet can be reached by means of ladder operators:

$|\mathbf{10}_4\rangle = \hat{T}_+|ddd\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_4\rangle = \frac{1}{\sqrt{3}}(|udd\rangle + |dud\rangle + |ddu\rangle)}$ $\quad |\mathbf{10}_5\rangle = \hat{T}_-|uuu\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_5\rangle = \frac{1}{\sqrt{3}}(|duu\rangle + |udu\rangle + |uud\rangle)}$

$|\mathbf{10}_6\rangle = \hat{U}_-|ddd\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_6\rangle = \frac{1}{\sqrt{3}}(|sdd\rangle + |dsd\rangle + |dds\rangle)}$ $\quad |\mathbf{10}_7\rangle = \hat{V}_-|uuu\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_7\rangle = \frac{1}{\sqrt{3}}(|suu\rangle + |usu\rangle + |uus\rangle)}$

$|\mathbf{10}_8\rangle = \hat{U}_+|sss\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_8\rangle = \frac{1}{\sqrt{3}}(|dss\rangle + |sds\rangle + |ssd\rangle)}$ $\quad |\mathbf{10}_9\rangle = \hat{V}_+|sss\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_9\rangle = \frac{1}{\sqrt{3}}(|uss\rangle + |sus\rangle + |uus\rangle)}$

- Also, the center point of the decuplet $\mathbf{10_S}$ can be reached via ladder operators, e.g.:

$|\mathbf{10}_{10}\rangle = \hat{T}_+|\mathbf{10}_6\rangle = \hat{T}_+(|sdd\rangle + |dsd\rangle + |dds\rangle) \overset{norm}{\Longrightarrow} \boxed{|\mathbf{10}_{10}\rangle = \frac{1}{\sqrt{6}}(|usd\rangle + |uds\rangle + |sud\rangle + |dus\rangle + |sdu\rangle + |dsu\rangle)}$

- The first point of the (mixed symmetric) octet $\mathbf{8_{MS}}$ can be derived by orthogonality and normalization conditions.

e.g. left top point; Ansatz $|\mathbf{8}_1^{MS}\rangle = \alpha|dd\rangle|u\rangle + \beta\frac{|ud\rangle+|du\rangle}{\sqrt{2}}|d\rangle = \alpha|ddu\rangle + \frac{\beta}{\sqrt{2}}|udd\rangle + \frac{\beta}{\sqrt{2}}|dud\rangle$ ... (1)

Orthogonality: $\langle\mathbf{10}_4|\mathbf{8}_1^{MS}\rangle = 0 \overset{(1)}{\Rightarrow} \frac{\langle udd| + \langle dud| + \langle ddu|}{\sqrt{3}}\left(\alpha|ddu\rangle + \frac{\beta}{\sqrt{2}}|udd\rangle + \frac{\beta}{\sqrt{2}}|dud\rangle\right) = 0 \Rightarrow \alpha = -\sqrt{2}\beta \overset{(1)}{\Rightarrow} \beta = \frac{1}{\sqrt{3}}$

Normalization: $\langle\mathbf{8}_1|\mathbf{8}_1\rangle = 1 \Rightarrow \boxed{|\mathbf{8}_1^{MS}\rangle = \frac{1}{\sqrt{6}}(|udd\rangle + |dud\rangle - 2|ddu\rangle)}$

- The other five border points of the octet $\mathbf{8_{MS}}$ can be reached by means of ladder operators:

$|\mathbf{8}_2^{MS}\rangle = \hat{T}_+|\mathbf{8}_1\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_2^{MS}\rangle = \frac{1}{\sqrt{6}}(2|uud\rangle - |udu\rangle - |duu\rangle)}$ $\quad |\mathbf{8}_3^{MS}\rangle = \hat{U}_-|\mathbf{8}_2\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_3^{MS}\rangle = \frac{1}{\sqrt{6}}(2|uus\rangle - |usu\rangle - |suu\rangle)}$

$|\mathbf{8}_4^{MS}\rangle = \hat{V}_-|\mathbf{8}_3\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_4^{MS}\rangle = \frac{1}{\sqrt{6}}(|sus\rangle + |uss\rangle - 2|ssu\rangle)}$ $\quad |\mathbf{8}_5^{MS}\rangle = \hat{T}_-|\mathbf{8}_4\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_5^{MS}\rangle = \frac{1}{\sqrt{6}}(|dss\rangle + |sds\rangle - 2|ssd\rangle)}$

$|\mathbf{8}_6^{MS}\rangle = \hat{U}_+|\mathbf{8}_5\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_6^{MS}\rangle = \frac{1}{\sqrt{6}}(2|dds\rangle - |dsd\rangle - |sdd\rangle)}$

- The first center point of the octet $\mathbf{8_{MS}}$ can also be reached by means of ladder operators, e.g.:

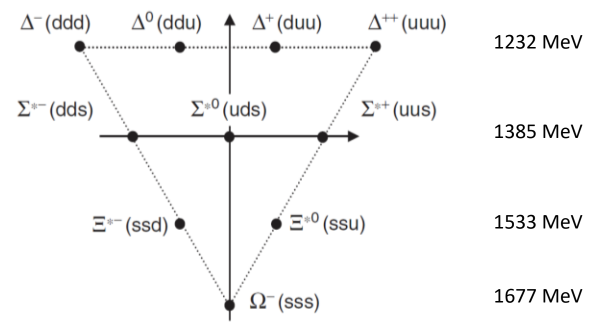
$|\mathbf{8}_7^{MS}\rangle = \hat{T}_+|\mathbf{8}_6^{MS}\rangle \overset{norm}{\Longrightarrow} \boxed{|\mathbf{8}_7^{MS}\rangle = \frac{1}{\sqrt{12}}(2|uds\rangle + 2|dus\rangle - |usd\rangle - |sud\rangle - |dsu\rangle - |sdu\rangle)}$

- Derivation of the second center state of the octet $\mathbf{8_{MS}}$ by further ladder operators, orthogonality and normalization:

Ansatz: $|\mathbf{8}_8^{MS}\rangle = \alpha\hat{U}_+|\mathbf{8}_4^{MS}\rangle + \beta\hat{V}_+|\mathbf{8}_5^{MS}\rangle$, orthogonality: $\langle\mathbf{8}_7^{MS}|\mathbf{8}_8^{MS}\rangle = 0 \overset{Norm}{\Longrightarrow} \boxed{|\mathbf{8}_8^{MS}\rangle = \frac{1}{2}(|sud\rangle + |usd\rangle - |dsu\rangle - |sdu\rangle)}$

observed $\mathbf{10_S}$ spin $\frac{3}{2}$ decuplet with $\Psi_A = \phi_S\chi_S\xi_A\eta_S$

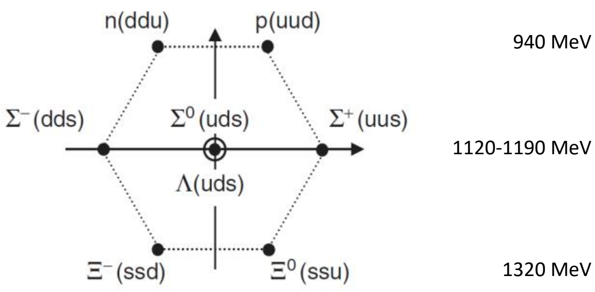
$L = 0, P = 1, S = \frac{3}{2}, J = \frac{3}{2}$



masses are not equal, because the s quark is heavier than u and d

observed $\mathbf{8_{MS}}$ spin $\frac{1}{2}$ octet with $\Psi_A = \underbrace{\phi_{MS}\chi_{MS}}_{sym.}\xi_A\eta_S$

$L = 0, P = 1, S = \frac{3}{2}, J = \frac{1}{2}$



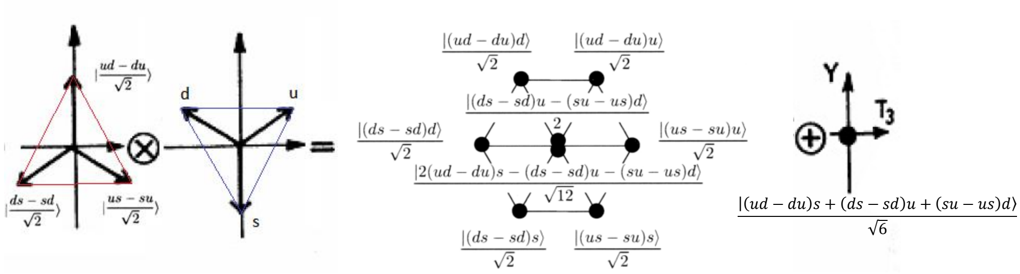
masses are not equal, because the s quark is heavier than u and d

Baryon masses:	$m = m_1 + m_2 + m_3 + A'\left(\frac{\langle\hat{S}_1\cdot\hat{S}_2\rangle}{m_1 m_2} + \frac{\langle\hat{S}_1\cdot\hat{S}_3\rangle}{m_1 m_3} + \frac{\langle\hat{S}_2\cdot\hat{S}_3\rangle}{m_2 m_3}\right)$	Effective masses:	$m_d = m_u = 0.365 GeV$ $m_s = 0.540\ GeV$	spin-spin interaction strength	$A' = 0.026 GeV^3$

Mixed Antisymmetric qqq Octet and Totally Antisymmetric qqq Singlet

- As described above: coupling $\mathbf{\overline{3}}_A \otimes \mathbf{3}$ (9 states) results into a six-sided (mixed antisymmetric) octet $\mathbf{8}_{MA}$ and a (antisymmetric) singlet. Hence: $\mathbf{\overline{3}}_A \otimes \mathbf{3} = \mathbf{8}_{MA} \oplus \mathbf{1}$

$$\mathbf{\overline{3}}_A \quad \otimes \quad \mathbf{3} \quad = \quad \mathbf{8}_{MA} \quad \oplus \quad \mathbf{1}_A$$



- The six corner-points of the octet $\mathbf{8}_{MA}$ can be read off directly:

$$|\mathbf{8}_1^{MA}\rangle = \frac{|ud\rangle-|du\rangle}{\sqrt{2}}|d\rangle = \frac{|udd\rangle-|dud\rangle}{\sqrt{2}} \qquad |\mathbf{8}_2^{MA}\rangle = \frac{|ud\rangle-|du\rangle}{\sqrt{2}}|u\rangle = \frac{|udu\rangle-|duu\rangle}{\sqrt{2}} \qquad |\mathbf{8}_3^{MA}\rangle = \frac{|us\rangle-|su\rangle}{\sqrt{2}}|u\rangle = \frac{|usu\rangle-|suu\rangle}{\sqrt{2}}$$

$$|\mathbf{8}_4^{MA}\rangle = \frac{|us\rangle-|su\rangle}{\sqrt{2}}|s\rangle = \frac{|uss\rangle-|sus\rangle}{\sqrt{2}} \qquad |\mathbf{8}_5^{MA}\rangle = \frac{|ds\rangle-|sd\rangle}{\sqrt{2}}|s\rangle = \frac{|dss\rangle-|sds\rangle}{\sqrt{2}} \qquad |\mathbf{8}_6^{MA}\rangle = \frac{|ds\rangle-|sd\rangle}{\sqrt{2}}|d\rangle = \frac{|dsd\rangle-|sdd\rangle}{\sqrt{2}}$$

- The first point in the center of the octet $\mathbf{8}_{MA}$ can be reached via ladder operators. As the isospin symmetry works best with T_3, we choose

$$|\mathbf{8}_7\rangle = \hat{T}_+|\mathbf{8}_6\rangle = \hat{T}_+\frac{|udd\rangle-|dud\rangle}{\sqrt{2}} = \hat{T}_+^{(1)}\frac{|dsd\rangle-|sdd\rangle}{norm} + \hat{T}_+^{(2)}\frac{|dsd\rangle-|sdd\rangle}{norm} + \hat{T}_+^{(3)}\frac{|dsd\rangle-|sdd\rangle}{norm} = \frac{|usd\rangle-0}{norm} + \frac{0-|sud\rangle}{norm} + \frac{|dsu\rangle-|sdu\rangle}{norm} \Longrightarrow$$

$$\boxed{|\mathbf{8}_7^{MA}\rangle = \frac{|dsu\rangle-|sdu\rangle+|usd\rangle-|sud\rangle}{2}}$$

- For the second point in the center of the octet $\mathbf{8}_{MA}$ we choose the following ansatz:

$$|\mathbf{8}_8^{MA}\rangle = \tilde{\alpha}\hat{U}_+|\mathbf{8}_4^{MA}\rangle + \tilde{\beta}\hat{V}_+|\mathbf{8}_5^{MA}\rangle = \tilde{\alpha}\hat{U}_+\frac{|uss\rangle-|sus\rangle}{\sqrt{2}} + \tilde{\beta}\hat{V}_+\frac{|dss\rangle-|sds\rangle}{\sqrt{2}}$$

$$|\mathbf{8}_8^{MA}\rangle = \alpha(-|dus\rangle+|uds\rangle+|usd\rangle-|sud\rangle) + \beta(-|uds\rangle+|dus\rangle+|dsu\rangle-|sdu\rangle) \ \ldots(1) \ \text{Orthogonality: } \langle\mathbf{8}_7^{MA}|\mathbf{8}_8^{MA}\rangle = 0 \Longrightarrow$$

$$\frac{\langle dsu|-\langle sdu|+\langle usd|-\langle sud|}{2}(-\alpha|dus\rangle+\alpha|uds\rangle+\alpha|usd\rangle-\alpha|sud\rangle-\beta|uds\rangle+\beta|dus\rangle+\beta|dsu\rangle-\beta|sdu\rangle) = 0$$

$$\frac{1}{2}\beta\langle dsu|dsu\rangle+\frac{1}{2}\beta\langle sdu|sdu\rangle+\frac{1}{2}\alpha\langle usd|usd\rangle+\frac{1}{2}\alpha\langle sud|sud\rangle = 0 \Longrightarrow \beta+\alpha = 0 \Longrightarrow \beta = -\alpha \overset{(1)}{\Longrightarrow}$$

$$|\mathbf{8}_8^{MA}\rangle = \alpha(2|uds\rangle-2|dus\rangle+|usd\rangle-|sud\rangle+|sdu\rangle-|dsu\rangle) \ \ldots(2) \ \text{Normalization: } \langle\mathbf{8}_8^{MA}|\mathbf{8}_8^{MA}\rangle = 1 \overset{(2)}{\Longrightarrow}$$

$$\alpha^2(4+4+1+1+1+1) = 12\alpha^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{12}} \overset{(2)}{\Longrightarrow} \boxed{|\mathbf{8}_8^{MA}\rangle = \frac{2(|uds\rangle-|dus\rangle)+|usd\rangle-|sud\rangle+|sdu\rangle-|dsu\rangle}{\sqrt{12}}}$$

- For the singlet $\mathbf{1}$ we choose the following ansatz: For each corner-point oft a (hypothetical) qq state $\mathbf{\overline{3}}_A$ there is a state of the third quark of the $\mathbf{3}$ triplet, which leads back to the center. We choose a superposition of these tree combinations, and also consider orthogonality to $|\mathbf{8}_7^{MA}\rangle$ and $|\mathbf{8}_8^{MA}\rangle$, and the normalization condition:

Ansatz: $|\mathbf{1}^A\rangle = \alpha\frac{|ud\rangle-|du\rangle}{\sqrt{2}}|s\rangle + \beta\frac{|ds\rangle-|sd\rangle}{\sqrt{2}}|u\rangle + \gamma\frac{|us\rangle-|su\rangle}{\sqrt{2}}|d\rangle = \alpha\frac{|uds\rangle-|dus\rangle}{\sqrt{2}} + \beta\frac{|dsu\rangle-|sdu\rangle}{\sqrt{2}} + \gamma\frac{|usd\rangle-|sud\rangle}{\sqrt{2}}$ ... (3)

Orthogonality 1: $\langle\mathbf{8}_7^{MA}|\mathbf{1}^A\rangle = 0 \Longrightarrow \frac{\langle dsu|-\langle sdu|+\langle usd|-\langle sud|}{2}\left(\alpha\frac{|uds\rangle-|dus\rangle}{\sqrt{2}} + \beta\frac{|dsu\rangle-|sdu\rangle}{\sqrt{2}} + \gamma\frac{|usd\rangle-|sud\rangle}{\sqrt{2}}\right) = 0$

$$\frac{1}{2}\frac{\beta}{\sqrt{2}}\langle dsu|dsu\rangle + \frac{1}{2}\frac{\beta}{\sqrt{2}}\langle sdu|sdu\rangle + \frac{1}{2}\frac{\gamma}{\sqrt{2}}\langle usd|usd\rangle + \frac{1}{2}\frac{\gamma}{\sqrt{2}}\langle sud|sud\rangle = 0 \Longrightarrow \beta+\gamma = 0 \Longrightarrow \gamma = -\beta \overset{(3)}{\Longrightarrow}$$

$$|\mathbf{1}\rangle = \alpha\frac{|uds\rangle-|dus\rangle}{\sqrt{2}} + \beta\frac{|dsu\rangle-|sdu\rangle}{\sqrt{2}} - \beta\frac{|usd\rangle-|sud\rangle}{\sqrt{2}} = \alpha\frac{|uds\rangle-|dus\rangle}{\sqrt{2}} + \beta\frac{|dsu\rangle-|sdu\rangle-|usd\rangle+|sud\rangle}{\sqrt{2}} \ \ldots(4)$$

Orthogonality 2: $\langle\mathbf{8}_8^{MA}|\mathbf{1}^A\rangle = 0 \Longrightarrow \frac{2\langle uds|-2\langle dus|+\langle usd|-\langle sud|+\langle sdu|-\langle dsu|}{2}\left(\alpha\frac{|uds\rangle-|dus\rangle}{\sqrt{2}} + \beta\frac{|dsu\rangle-|sdu\rangle-|usd\rangle+|sud\rangle}{\sqrt{2}}\right) = 0$

$$\frac{2}{\sqrt{12}}\frac{\alpha}{\sqrt{2}}\langle uds|uds\rangle + \frac{2}{\sqrt{12}}\frac{\alpha}{\sqrt{2}}\langle dus|dus\rangle - \frac{1}{\sqrt{12}}\frac{\beta}{\sqrt{2}}\langle usd|usd\rangle - \frac{1}{\sqrt{12}}\frac{\beta}{\sqrt{2}}\langle sud|sud\rangle - \frac{1}{\sqrt{12}}\frac{\beta}{\sqrt{2}}\langle sdu|sdu\rangle - \frac{1}{\sqrt{12}}\frac{\beta}{\sqrt{2}}\langle dsu|dsu\rangle = 0$$

$4\alpha-4\beta = 0 \Longrightarrow \beta = \alpha \overset{(4)}{\Longrightarrow} |\mathbf{1}^A\rangle = \alpha\frac{|uds\rangle-|dus\rangle+|dsu\rangle-|sdu\rangle-|usd\rangle+|sud\rangle}{\sqrt{2}}$ ... (5) Normalization: $\langle\mathbf{1}|\mathbf{1}\rangle = 1 \overset{(5)}{\Longrightarrow} \frac{\alpha^2}{2}(1+1+1+1+1+1) = 1$

$$3\alpha^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{3}} \overset{(5)}{\Longrightarrow} \boxed{|\mathbf{1}^A\rangle = \frac{1}{\sqrt{6}}(|uds\rangle-|dus\rangle+|dsu\rangle-|sdu\rangle-|usd\rangle+|sud\rangle)}$$

The Local Gauge Principle

QED U(1)

- The free particle Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ is invariant under a *global* phase transformation $\psi(x^\alpha) \to \psi'(x^\alpha) = e^{i\varphi}\,\psi(x^\alpha)$
- Suppose we now demand invariance of physics under a *local* phase transformation
  $\psi(x^\alpha) \to \psi'(x^\alpha) = \hat{U}(x^\alpha)\,\psi(x^\alpha) = e^{iq\chi(x^\alpha)}\,\psi(x^\alpha)$
- The free particle Dirac equation is not invariant under this *local* transformation: $i\gamma^\mu\partial_\mu\left(e^{iq\chi(x^\alpha)}\psi\right) = me^{iq\chi(x^\alpha)}\psi \Longrightarrow$
  $i\gamma^\mu\left(\partial_\mu e^{iq\chi(x^\alpha)}\psi + e^{iq\chi(x^\alpha)}\partial_\mu\psi\right) = me^{iq\chi(x^\alpha)}\psi \Longrightarrow i\gamma^\mu\left(e^{iq\chi(x^\alpha)}iq\partial_\mu\chi(x^\alpha)\,\psi + e^{iq\chi(x^\alpha)}\partial_\mu\psi\right) = me^{iq\chi(x^\alpha)}\psi \Longrightarrow$
  $\boxed{i\gamma^\mu\left(\partial_\mu + iq\partial_\mu\chi(x^\alpha)\right)\psi = m\psi}$. This equation differs from the original equation by the term $-q\partial_\mu\chi(x^\alpha)$
- Whenever a physical theory with some *global* transformation invariance should also become invariant under the same *local* transformation, it is necessary to introduce new fields.
- In case of the Dirac equation we need to introduce the EM field by means of the four-potential $A^\mu(x^\alpha) = \left(\phi(x^\alpha), \vec{A}(x^\alpha)\right)^T$
- The physics of the EM field does not change under the *global* gauge transformation $A_\mu(x^\alpha) \to A'_\mu(x^\alpha) = A_\mu(x^\alpha) - \partial_\mu\chi$
  where $A_\mu(x^\alpha) = \left(\phi(x^\alpha), -\vec{A}(x^\alpha)\right)^T$ and $\partial_\mu = \left(\partial_0, \vec{\nabla}\right)^T$.
- Again, we demand invariance under *local* gauge transformation $\boxed{A_\mu(x^\alpha) \to A'_\mu(x^\alpha) = A_\mu(x^\alpha) - \partial_\mu\chi(x^\alpha)}$
- When we demand invariant physics under the local gauge transformation $\boxed{\psi(x^\alpha) \to \psi'(x^\alpha) = e^{iq\chi(x^\alpha)}\,\psi(x^\alpha)}$, we can achieve this goal by re-writing the free-particle Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ to $\to$ $\boxed{i\gamma^\mu\underbrace{\left(\partial_\mu + iqA_\mu(x^\alpha)\right)}_{D_\mu\text{ min. substitution}}\psi = m\psi}$
- This modified Dirac equation no longer corresponds to a wave equation for a free particle, because there is now an **interaction term** $\boxed{-q\gamma^\mu A_\mu(x^\alpha)\,\psi}$.

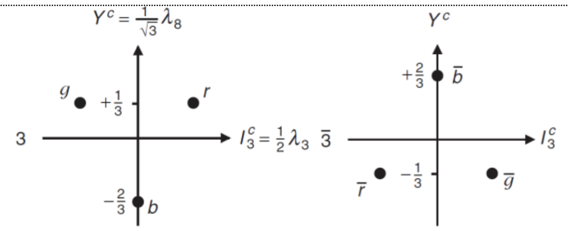
QCD SU(3)

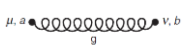
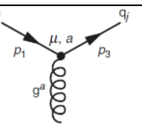
- The corresponding SU(3) symmetry associated with QCD is invariance under the following local gauge transformation:
  $\boxed{\psi(x^\alpha) \to \psi'(x^\alpha) = e^{ig_s\vartheta(x^\alpha)_a\hat{T}_a}\,\psi(x^\alpha)}$ with $a = 1\ldots8$, $\hat{T}_a$ being the eight generators of the SU(3) symmetry group related to the eight Gell-Mann-Matrices by $\hat{T}_a = \frac{1}{2}\hat{\lambda}_a$, and $\vartheta(x^\alpha)_a$ being eight functions of the space-time coordinate x^α
- With this local gauge transformation, the free-particle Dirac equation becomes $\boxed{i\gamma^\mu\left(\partial_\mu + ig_S\partial_\mu\vartheta(x^\alpha)_a\hat{T}_a\right)\psi = m\psi}$
- The required local gauge invariance can be asserted by introducing eight new fields $G_\mu(x^\alpha)_a$, representing the gluons
- These 8 new fields transform in a *local* gauge transformation $\boxed{G_\mu(x^\alpha)_k \to G'_\mu(x^\alpha)_k = G_\mu(x^\alpha)_k - \partial_\mu\vartheta(x^\alpha)_k - g_Sf_{ijk}\vartheta(x^\alpha)_i\,G_\mu(x^\alpha)_j}$
- The last term arises because the SU(3) generators do not commute $\left[\hat{\lambda}_a, \hat{\lambda}_b\right] = 2if_{abc}\hat{\lambda}_c$.
- The presence of the last term gives rise to gluon self-interactions.
- The Dirac equation, including the interactions with the new gauge fields, becomes $\boxed{i\gamma^\mu\left(\partial_\mu + ig_S\,G_\mu(x^\alpha)_a\hat{T}_a\right)\psi = m\psi}$
- From this we see that the form of the qqg **interaction term** is $\boxed{-g_S\gamma^\mu\,G_\mu(x^\alpha)_a\,T_a = -\frac{1}{2}g_S\gamma^\mu\,G_\mu(x^\alpha)_a\,\hat{\lambda}_a}$
- These are the structure constants and their properties: $f_{123} = 1$; $f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}$; $f_{458} = f_{678} = \frac{\sqrt{3}}{2}$;
  $f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{acb} = -f_{cba}$ (anti-symmetric). All other structure constants are zero.

Color and QCD

- Instead of two types of charge like in electromagnetism, QCD has three fundamental charges for the strong nuclear force, known as colors, called red, green and blue (r, g, b) and three opposite color charges anti-red, anti-green and anti-blue $(\bar{r}, \bar{g}, \bar{b})$.
- A free particle cannot have a net charge of any type; only "colorless" states are allowed for free (observable) particles.
- A color plus its anti-color is colorless; additionally, all three unique colors r, g, b (or anticolors $\bar{r}, \bar{g}, \bar{b}$) added together are colorless
- Whereas the QED interaction is mediated by a massless photon corresponding to the single generator of the U(1) local gauge symmetry, the QCD interaction is mediated by eight massless gluons corresponding to the eight generators of the SU(3) local gauge symmetry.
- Each quark contains a net color charge of one color (r, g, b); each antiquark has an anti-color $(\bar{r}, \bar{g}, \bar{b})$ assigned to it
- The only other Standard Model particle with a color is the gluon: quarks exchange gluons, and that's how they form bound states.
- Only particle with a non-zero color charge couple to gluons. For this reason, leptons do not feel the strong force.
- The quarks, which carry the color charge, exist in three orthogonal color states.
- Unlike the approximate SU(3) uds flavor symmetry, the SU(3) rgb color symmetry is exact. Consequently, the strength of the QCD interaction is independent of the color charge.
- Whereas electromagnetism doesn't change the electric charge of the particles attracting or repelling one another, the colors (or anticolors) of the quarks (or antiquarks) change every time the strong nuclear force occurs.

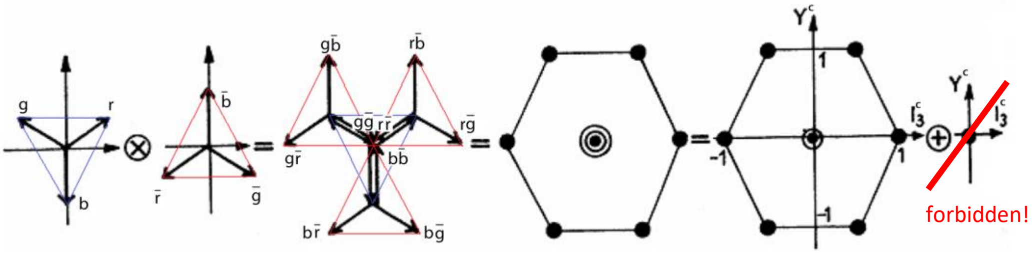
- The 3 color states can be represented by 3 color wave functions
  $$\mathrm{r} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathrm{g} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathrm{b} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$
- Following the discussion of SU(3) flavor, the color states of quarks and antiquarks can be labelled by two quantum numbers: The third component of **color isospin** $\mathbf{I}_3^c$ and the **color hypercharge** $\mathbf{Y}^c$



Feynman Rules for QCD	Gluon propagator	$-\dfrac{ig_{\mu\nu}}{q^\alpha q_\alpha}\delta^{ab}$		δ^{ab} ensures that the gluon of type a emitted at vertex μ is the same as absorbed at ν	QCD vertex:	$-\frac{1}{2}ig_S\lambda^a_{ji}\gamma^\mu$ $a = 1\ldots8$ $ji = r, g, b$	

Gluons: Coupling color and anticolor to a $c\bar{c}$ octet (and a forbidden $c\bar{c}$ singlet)

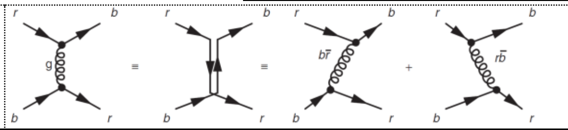
- By coupling a color triplet $\mathbf{3}$ with an anticolor triplet $\bar{\mathbf{3}}$, altogether $\mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{9}$ color-anticolor states (i.e. gluons) could be theoretically produced. These 9 states split into an octet and a singlet: $\mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, as is derived below.
- The singlet is forbidden for gluons though, as a single gluon (which is not observable in a free state) is not allowed to be colorless! Therefore, only 8 gluons exist.
- Graphical derivation: The vertices of the color triplet $\mathbf{3}$ are the center-points for the three coupled anti-triplets $\bar{\mathbf{3}}$. This creates a hexagon with six uniquely occupied corners and a triple-occupied center.



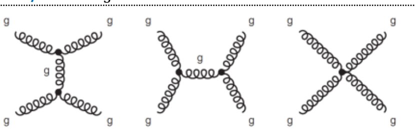
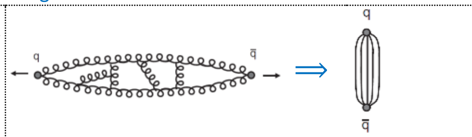
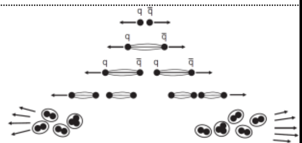
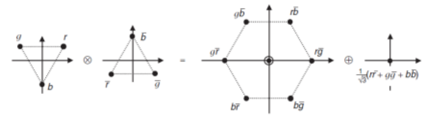
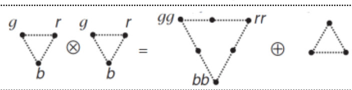
- In a SU (3) multiplet, each point of the outer shell is occupied once, and each point of the next inner shell (if the outer shell was not a triangle) is occupied twice. The outer shell here is a hexagon; the next inner shell is the center, which should therefore only be occupied twice (instead of three-fold). Therefore, the hexagon with the triple-occupied center can be decomposed into a hexagon with a double-occupied center (an octet) and a singlet (the additionally occupied center point).
- The six border states $\xi_8(I^c, I_3^c, Y^c)$ of the octet are unique (quantum number $I^c = (I_3^c)_{max} = 1$):

$$\boxed{G_8\left(1,-\tfrac{1}{2},+1\right) = |g\bar{b}\rangle, G_8\left(1,+\tfrac{1}{2},+1\right) = |r\bar{b}\rangle, G_8(1,+1,0) = |r\bar{g}\rangle, G_8\left(1,+\tfrac{1}{2},-1\right) = |b\bar{g}\rangle, G_8\left(1,-\tfrac{1}{2},-1\right) = |b\bar{r}\rangle, G_8\left(1,-\tfrac{1}{2},0\right) = |g\bar{r}\rangle}$$

- The first center state $G_8^{C1}(1,0,0)$ can be derived by using ladder operators.
  We choose $G_8^{C1}(0,0,1) = \hat{T}_-^c|r\bar{g}\rangle = \hat{T}_-^{(1)}|r\bar{g}\rangle + \hat{\bar{T}}_-^{(2)}|r\bar{g}\rangle \overset{norm}{\Longrightarrow} \boxed{G_8^{C1}(1,0,0) = \frac{1}{\sqrt{2}}(|r\bar{r}\rangle - |g\bar{g}\rangle)}$ ... (1)
- The second center state $G_8^{C2}(1,0,0)$ can be derived by means of further ladder operators:
  $\hat{V}_-^c|r\bar{s}\rangle \overset{norm}{=} \frac{1}{\sqrt{2}}(|b\bar{b}\rangle - |r\bar{r}\rangle)$ ... (2) $\hat{U}_-|g\bar{b}\rangle \overset{norm}{=} \frac{1}{\sqrt{2}}(|b\bar{b}\rangle - |g\bar{g}\rangle)$ ... (3) Superposition: $G_8^{C2}(0,0,1) = \alpha\frac{|b\bar{b}\rangle - |r\bar{r}\rangle}{\sqrt{2}} + \beta\frac{|b\bar{b}\rangle - |g\bar{g}\rangle}{\sqrt{2}}$ ... (4)
  with : $\langle C1|C2\rangle = 0$ and $\alpha^2 + \beta^2 = 1 \Longrightarrow \boxed{G_8^{C2}(1,0,0) = \frac{1}{\sqrt{6}}\left(|r\bar{r}\rangle + |g\bar{g}\rangle - 2|b\bar{b}\rangle\right)}$

Color flow for the t-channel process rb→br		Eight combinations: $\frac{r\bar{b}+b\bar{r}}{\sqrt{2}}, \frac{r\bar{g}+g\bar{r}}{\sqrt{2}}, \frac{b\bar{g}+g\bar{b}}{\sqrt{2}}, \frac{r\bar{r}-b\bar{b}}{\sqrt{2}}, -i\frac{r\bar{b}-b\bar{r}}{\sqrt{2}}, -i\frac{r\bar{g}-g\bar{r}}{\sqrt{2}}, -i\frac{b\bar{g}+g\bar{b}}{\sqrt{2}}, \frac{r\bar{r}+b\bar{b}-2g\bar{g}}{\sqrt{2}}$

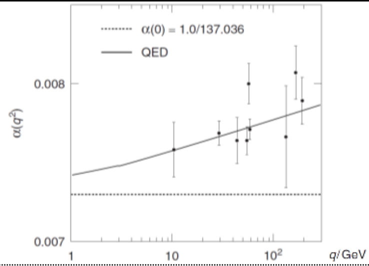
Color Confinement

Color confinement hypothesis	Colored objects are always confined to colorless singlet states. No object with nonzero color can propagate as free particle. Therefore we never see free quarks. They are always confined to bound colorless states. Gluons, being colored, are also confined to colorless objects. Therefore gluons do not propagate over macroscopic distances.	
colorless	Only when $I_3^c = Y^c = 0$ a state is colorless. This is only the case for color singlet states.	
gg interaction triple and quartic gluon vertices	 Virtual gluons carry color and interact with each other: Therefore they form flux-tubes  Energy stored in the field is proportional to the separation with $1GeV/fm$! This is $10^5 N$ between any 2 unconfined quarks!	
Hadronic states	The color confinement hypothesis implies that all hadrons (bound quark states) are colorless, i.e. have a colorless, singlet color state. This strongly restricts the possible quark combinations. Not to be confused: (Single) gluon states are <u>not</u> allowed to be (colorless) singlet color states. All free (observable) colored objects <u>must be</u> bound (colorless) singlet color states	
Hadronisation	In a process like $e^+e^- \to q\bar{q}$ two (initially free) quarks are produced traveling back to back. As they separate the color field is restricted to a tube. When the energy stored in the color field is sufficient, the tube breaks into smaller "strings" producing new $q\bar{q}$ pairs. This results in a jet of hadrons (bound quark states).	
allowed $q\bar{q}$ meson state	The only possible singlet color state for $(q\bar{q})$ mesons is $\xi_c(q\bar{q}) = \frac{1}{\sqrt{3}}(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle)$	
forbidden qq meson state	There is no singlet color wavefunction for (qq) combinations, therefore there are no qq meson states	
allowed qqq baryon state	The only possible totally antisymmetric singlet color state for qqq mesons is $\xi_c(qqq) = \frac{1}{\sqrt{6}}(|rgb\rangle - |grb\rangle + |gbr\rangle - |bgr\rangle - |rbg\rangle + |brg\rangle)$	$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}_S \oplus \bar{\mathbf{3}}_A)\otimes\mathbf{3} = \mathbf{6}_S\otimes\mathbf{3} \oplus \bar{\mathbf{3}}_A\otimes\mathbf{3} = (\mathbf{10}_S \oplus \mathbf{8}_{MS}) \oplus (\mathbf{8}_{MA} \oplus \mathbf{1}_A)$
other baryon states	Another possible and confirmed state is the antibaryon $(\bar{q}\bar{q}\bar{q})$ state. Pentaquark states $(qqqq\bar{q})$ were observed in 2015 and 2019 by LHCb in CERN. In principle, also other combinations of $(q\bar{q})$ and (qqq) could exist.	

Running Coupling Constant in QED

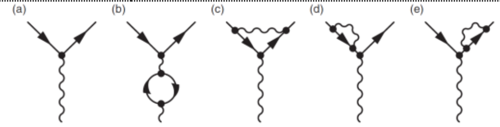
General

The coupling constant of QED at low energies is small ($\alpha \sim 1/137$). Therefore, first order calculations already yield good results, and perturbation theory does work well. However, it must be taken into account that in QED α is not constant, but becomes <u>larger</u> at higher energies ("running coupling constant").



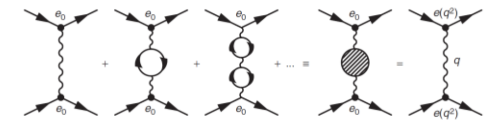
Ward identity

In a field theory with local gauge invariance, higher-order corrections to four-vector currents as shown in (c), (d) and (d) cancel each other out. Only loop corrections to the photon propagator as shown for first order in (b) must be considered.



Photon self-energy

The infinite series of corrections to the photon propagator, known as the photon self-energy terms, are accounted for by replacing the lowest-order photon exchange diagram by the infinite series of loop diagrams expressed in terms of the bare electron charge e_0. The corrections are absorbed into the charge $e_0 \rightarrow e(q^\alpha q_\alpha)$



Calulating the running coupling constant $\alpha(q^2)$ with renormalization

Each loop introduces a correction factor $\pi(q^2)$ such that the effective propagator P is given by:

$$P = P_0 + P_0\,\pi(q^2)\,P_0 + P_0\,\pi(q^2)\,P_0\,\pi(q^2)\,P_0 + \cdots = P_0(P_0\,\pi(q^2) + P_0^2\,\pi(q^2)^2 + \cdots) \Longrightarrow P = P_0\frac{1}{1-P_0\,\pi(q^2)} \ \dots (1)$$

One-loop photon self-energy correction: $\Pi(q^2) = \frac{\pi(q^2)}{q^2} \Longrightarrow \pi(q^2) = q^2\,\Pi(q^2)$ … (2) $\overset{(1)}{\Longrightarrow}$

$$P = P_0\frac{1}{1-P_0 q^2\,\Pi(q^2)} \ \dots (3)$$ Propagator with bare charge: $P_0 = \frac{e_0^2}{q^2} \overset{(3)}{\Longrightarrow} P = P_0\frac{1}{1-e_0^2\,\Pi(q^2)}$ … (4)

We want to express the effective propagator P in terms of the running coupling $e(q^2)$:

$$P \overset{\text{def}}{=} \frac{e^2(q^2)}{q^2} \overset{(4)}{\Longrightarrow} P_0\frac{1}{1-e_0^2\,\Pi(q^2)} = \frac{e^2(q^2)}{q^2}\ \Big|\ P_0 = \frac{e_0^2}{q^2} \Longrightarrow \frac{e_0^2}{q^2}\frac{1}{1-e_0^2\,\Pi(q^2)} = \frac{e^2(q^2)}{q^2} \Longrightarrow e^2(q^2) = \frac{e_0^2}{1-e_0^2\,\Pi(q^2)} \ \dots (5)$$

Problem: $\Pi(q^2)$ is divergent. But we know from experiment that $e(q^2)$ is finite.

If we know the electron charge $e(q^2)$ at some scale $q^2 = \mu^2$, we can write: $\overset{(5)}{\Longrightarrow} e^2(\mu^2) = \frac{e_0^2}{1-e_0^2\,\Pi(\mu^2)} \Longrightarrow$

$$e^2(\mu^2) - e^2(\mu^2)\,e_0^2\,\Pi(\mu^2) = e_0^2 \Longrightarrow e_0^2 + e^2(\mu^2)\,e_0^2\,\Pi(\mu^2) = e^2(\mu^2) \Longrightarrow e_0^2(1 + e^2(\mu^2)\,\Pi(\mu^2)) = e^2(\mu^2) \Longrightarrow$$

$$e_0^2 = \frac{e^2(\mu^2)}{1+e^2(\mu^2)\,\Pi(\mu^2)} \overset{(5)}{\Longrightarrow} e^2(q^2) = \frac{\frac{e^2(\mu^2)}{1+e^2(\mu^2)\Pi(\mu^2)}}{1-\frac{e^2(\mu^2)}{1+e^2(\mu^2)\Pi(\mu^2)}\Pi(q^2)} = \frac{e^2(\mu^2)}{1+e^2(\mu^2)\,\Pi(\mu^2)-e^2(\mu^2)\,\Pi(q^2)} \Longrightarrow e^2(q^2) = \frac{e^2(\mu^2)}{1+e^2(\mu^2)(\Pi(\mu^2)-\Pi(q^2))} \Longrightarrow$$

$$e^2(q^2) = \frac{e^2(\mu^2)}{1-e^2(\mu^2)(\Pi(q^2)-\Pi(\mu^2))} \ \dots (6)$$ Both $\Pi(q^2)$ and $\Pi(\mu^2)$ are divergent, but $\Pi(q^2) - \Pi(\mu^2)$ is finite.

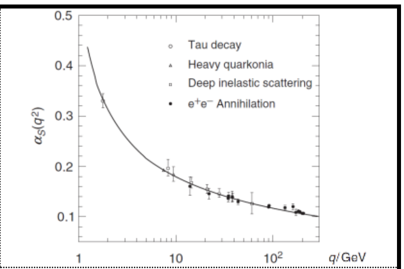
$$\Pi(q^2) - \Pi(\mu^2) \approx \frac{1}{12\pi^2}\ln\left(\frac{q^2}{\mu^2}\right) \overset{(6)}{\Longrightarrow} e^2(q^2) = \frac{e^2(\mu^2)}{1-e^2(\mu^2)\frac{1}{12\pi^2}\ln\left(\frac{q^2}{\mu^2}\right)}\ \Big|\ e^2(q^2) = 4\pi\,\alpha(q^2) \Longrightarrow 4\pi\,\alpha(q^2) = \frac{4\pi\,\alpha(\mu^2)}{1-\alpha(\mu^2)\frac{1}{3\pi}\ln\left(\frac{q^2}{\mu^2}\right)} \Longrightarrow$$

$$\boxed{\alpha(q^2) = \frac{\alpha(\mu^2)}{1-\alpha(\mu^2)\frac{1}{3\pi}\ln\left(\frac{q^2}{\mu^2}\right)}}$$ $\alpha(0) = \frac{1}{137.035\,999\,075}$ At center-of-mass energy $\sqrt{s} = 193\ GeV$: $\alpha = \frac{1}{127.4}$

Running Coupling Constant in QCD

General

The coupling constant of QCD at low energies is large ($\alpha_s \sim 1$). Therefore, first order calculations are not sufficient, and perturbation theory does not work. However, it must be taken into account that in QCD α_s is not constant, but becomes <u>smaller</u> at higher energies ("running coupling constant") so that perturbation theory can be used in the high-energy regime.

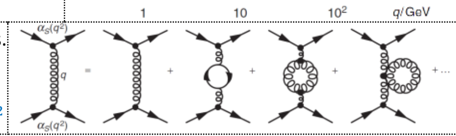


Gluon self energy

Owning to the gluon-gluon self-interaction, there are additional diagrams.

$\Pi(q^2) - \Pi(\mu^2) \approx -\frac{B}{4\pi}\ln\left(\frac{q^2}{\mu^2}\right)$ with $B = \frac{11N_c - 2N_f}{12\pi}$ N_f … #quark flavors, N_c … #of colors

For $N_c = 3$ and $N_f \leq 6$ quarks, $B > 0 \Longrightarrow \alpha_s$ decreases with increasing q^2



$\alpha_s(q^2)$

$$\boxed{\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1+B\,\alpha_s(\mu^2)\frac{1}{3\pi}\ln\left(\frac{q^2}{\mu^2}\right)} \text{ with } B = \frac{11N_c-2N_f}{12\pi}}$$

Asymptotic freedom: At $|q| > 100 GeV$ $\alpha_s \sim 0.1$. Quarks can be treated as quasi-free particles. Perturbation theory can be used.

Parity Conservation in QED and QCD, Parity Violation in Weak Interaction

Parity operator

$\hat P\,\Psi(\vec x,t)=\Psi(-\vec x,t)$, $\hat P\hat P\,\Psi(\vec x,t)=\Psi(\vec x,t)$ $\Longrightarrow$ $\hat P\hat P=\mathbb{1}$, $\hat P=\hat P^{-1}$

If physics is invariant under $\hat P$: $\hat P^\dagger\hat P=\mathbb{1}\Leftrightarrow\hat P^\dagger=\hat P^{-1}$... unitary, $\hat P=\hat P^{-1}\Longrightarrow\hat P^\dagger=\hat P$ ... Hermitian

If $[\hat H,\hat P]=0$ then $\hat P\Psi=\mathrm{P}\Psi$ with $\mathrm{P}=\pm1$ because $\hat P^2\Psi=\mathrm{P}^2\Psi=\Psi$

intrinsic parity

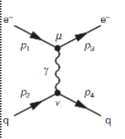
$\mathrm{P}(e^-)=\mathrm{P}(\nu_e)=\mathrm{P}(q)=+1$ | $\mathrm{P}(e^+)=\mathrm{P}(\bar\nu_e)=\mathrm{P}(\bar q)=-1$ | $\mathrm{P}(\gamma)=\mathrm{P}(g)=\mathrm{P}(W^\pm)=\mathrm{P}(Z)=-1$ | $\mathrm{P(Higgs)}=+1$

$\hat P$ in QED

$\hat P=\gamma^0\Longrightarrow U\xrightarrow{\hat P}\hat P U=\gamma^0U$

adjoint spinor $\bar U=U^\dagger\gamma^0\xrightarrow{\hat P}(\hat P U)^\dagger\gamma^0=(\gamma^0U)^\dagger\gamma^0=U^\dagger\gamma^{0\dagger}\gamma^0=U^\dagger\gamma^0\gamma^0=\bar U\gamma^0\Longrightarrow\bar U\xrightarrow{\hat P}\bar U\gamma^0$

QED matrix element



$j_e^\mu=\bar U_e(p_3^\sigma)\,\gamma^\mu\,U_e(p_1^\sigma)\xrightarrow{\hat P}\bar U_e(p_3^\sigma)\,\gamma^0\gamma^\mu\gamma^0\,U_e(p_1^\sigma)\Longrightarrow$

$j_e^0\xrightarrow{\hat P}\bar U_e(p_3^\sigma)\,\gamma^0\gamma^0\gamma^0\,U_e(p_1^\sigma)=\bar U_e(p_3^\sigma)\,\gamma^0\,U_e(p_1^\sigma)=j_e^0$

$j_e^k\xrightarrow{\hat P}\bar U_e(p_3^\sigma)\,\gamma^0\gamma^k\gamma^0\,U_e(p_1^\sigma)=-\bar U_e(p_3^\sigma)\,\gamma^k\gamma^0\gamma^0\,U_e(p_1^\sigma)=-\bar U_e(p_3^\sigma)\,\gamma^k\,U_e(p_1^\sigma)=-j_e^k$

$j_e^\mu j_\mu^q=j_e^0j_0^q-j_e^kj_k^q\xrightarrow{\hat P}j_e^0j_0^q-(-j_e^k)(-j_k^q)=j_e^\mu j_\mu^q\Longrightarrow$ invariant under $\hat P\Longrightarrow$ **parity is conserved in QED**

QCD

Apart from the color factors, the QCD interaction has the same form $\Longrightarrow$ invariant under $\hat P\Longrightarrow$ **parity is conserved in QCD**

P. conservation in decay

The total parity of a two-body final state is the product of the intrinsic parities and the parity of the orbital wavefunction, which is given by $(-1)^l$ with l being the orbital angular momentum in the final state.

Examples for allowed and forbidden deacys

Consider $\rho^0(1^-)\to\pi^++\pi^-$. In order to conserve angular momentum, the π^+ and π^- are produced with relative orbital angular momentum $l=1$. Therefore $\mathrm{P}(\rho^0)=\mathrm{P}(\pi^+)\,\mathrm{P}(\pi^-)\,(-1)^l\Longrightarrow-1=(-1)(-1)(-1)^1$ ✓allowed

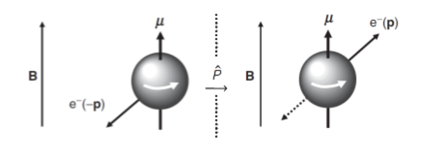
Consider $\rho^0(0^-)\to\pi^++\pi^-$. In order to conserve angular momentum, the π^+ and π^- are produced with relative orbital angular momentum $l=0$. Therefore $\mathrm{P}(\rho^0)=\mathrm{P}(\pi^+)\,\mathrm{P}(\pi^-)\,(-1)^l\Longrightarrow-1=(-1)(-1)(-1)^0$ ↯ forbidden

Parity properties

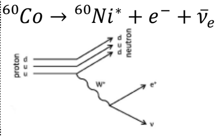
	Rank	Parity	Example	Lorentz-invariant bilinear covariant currents: Form	components	Boson spin
Scalar	0	+	temperature T	$j_S=\bar\psi\phi$	1	0
Pseudoscalar	0	-	helicity h	$j_{PS}=\bar\psi\gamma^5\phi$	1	0
Vector	1	-	momentum $\vec p$	$j_V^\mu=\bar\psi\gamma^\mu\phi$	4	1
Axial (pseudo)vector	1	+	angular momentum $\vec L$	$j_A^\mu=\bar\psi\gamma^\mu\gamma^5\phi$	4	1
Tensor	2			$j_T^{\mu\nu}=\bar\psi(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\phi$	6	2

$j_S\xrightarrow{\hat P}j_S$ $\quad j_{PS}\xrightarrow{\hat P}-j_{PS}$ $\quad j_V^0\xrightarrow{\hat P}j_V^0,\ j_V^k\xrightarrow{\hat P}-j_V^k\Longrightarrow j_V^\mu j_\mu^V\xrightarrow{\hat P}j_V^\mu j_\mu^V$ $\quad j_A^0\xrightarrow{\hat P}-j_A^0,\ j_A^k\xrightarrow{\hat P}j_A^k\Longrightarrow j_A^\mu j_\mu^A\xrightarrow{\hat P}j_A^\mu j_\mu^A$ $\quad j_A^\mu j_\mu^V\xrightarrow{\hat P}-j_A^\mu j_\mu^V$

Parity violation in β-decay of polarized cobalt-60.



$^{60}Co\to{}^{60}Ni^*+e^-+\bar\nu_e$

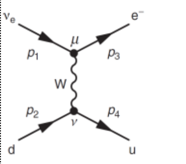


Experiment: The spin magnetic moment $\vec\mu$ of ^{60}Co atoms is aligned by an strong external magnetic field $\vec B$. It turns out that much more electrons from beta-decay are emitted opposite to the $\vec B$ direction (and hence **opposite to the spin direction** of the atoms)

Direct interpretation: The weak interaction “cares” about the spin direction (couples only to LH particles)

Parity violation interpretation: In a (hypothetically) experiment mirrored under parity-transformation $\hat P$, the axial vectors $\vec B$ and $\vec\mu$ do not change orientation. Only the vector momentum $\vec p_e$ of the emitted electrons changes sign. Therefore, in the $\hat P$-transformed “mirror-world”, the electrons would predominantly be emitted **into** $\vec B$ direction (and hence **into the spin direction** of the atoms). $\Longrightarrow$ The parity-transformed experiment would have a different outcome $\Longrightarrow$ parity violation.

Neutrino scattering



Inv. β-decay charged-current weak interaction $\nu_ed\to e^-u$:

$j_{\nu e}^\mu=\bar{\mathrm U}(p_3^\sigma)\,(g_V\gamma^\mu+g_A\gamma^\mu\gamma^5)\,\mathrm U(p_1^\sigma)=g_Vj_{\nu e}^{V\mu}+g_Aj_{\nu e}^{A\mu}$

$j_{du}^\nu=\bar{\mathrm U}(p_4^\sigma)\,(g_V\gamma^\nu+g_A\gamma^\nu\gamma^5)\,\mathrm U(p_2^\sigma)=g_Vj_{du}^{V\nu}+g_Aj_{du}^{A\nu}$

$\mathcal M_{fi}\propto j_{\nu e}^\mu j_{du\mu}^V=g_V^2j_{\nu e}^{V\mu}j_{du\mu}^V+g_A^2j_{\nu e}^{A\mu}j_{du\mu}^A+g_Vg_A(j_{\nu e}^{V\mu}j_{du\mu}^A+j_{\nu e}^{A\mu}j_{du\mu}^V)$

$j_{\nu e}^\mu j_{du\mu}^V\xrightarrow{\hat P}\underbrace{g_V^2j_{\nu e}^{V\mu}j_{du\mu}^V+g_A^2j_{\nu e}^{A\mu}j_{du\mu}^A}_{conserves\ parity}-\underbrace{g_Vg_A(j_{\nu e}^{V\mu}j_{du\mu}^A+j_{\nu e}^{A\mu}j_{du\mu}^V)}_{violates\ parity}$

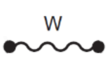
strength of parity violating part

$$\frac{g_Vg_A}{g_V^2+g_A^2}$$

max. violation when $|g_V|=|g_A|$

Weak Interaction: Feynman Rules and Chiral Structure. Strength of Weak Interaction.

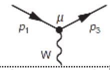
Propagator of massive W boson



$i\frac{1}{q^\alpha q_\alpha-m_w^2}\underbrace{\sum_\lambda\varepsilon_\mu^{\lambda*}\varepsilon_\nu^\lambda}_{polarization\ states}=\frac{i}{q^\alpha q_\alpha-m_w^2}\underbrace{\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_w^2}\right)}_{because\ massive\ W\ boson\ has\ also\ longitud.\ state}=\boxed{-\frac{i}{q^\alpha q_\alpha-m_w^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{m_w^2}\right)}$

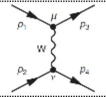
Propagator limit when $q^\alpha q_\alpha\ll m_w^2$: $-\frac{ig_{\mu\nu}}{q^\alpha q_\alpha-m_w^2}$

weak charged vertex factor



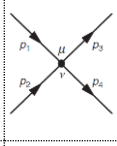
Weak charged current with $W^\pm$ bosons is a $(V-A)$ interaction with vertex-factor $\boxed{-i\frac{g_W}{\sqrt2}\frac{1}{2}\gamma^\mu(1-\gamma^5)}$

Matrix - element $\mathcal M_{fi}$



$\mathcal M_{fi}=-\left(\frac{g_W}{\sqrt2}\bar\Psi_3\frac12\gamma^\mu(1-\gamma^5)\Psi_1\right)\frac{g_{\mu\nu}-q_\mu q_\nu/m_W^2}{q^\alpha q_\alpha-m_W^2}\left(\frac{g_W}{\sqrt2}\bar\Psi_4\frac12\gamma^\nu(1-\gamma^5)\Psi_2\right)$

Fermi Theory $\mathcal M_{fi}$ limit $q^\alpha q_\alpha\ll m_w^2$



ansatz: $\boxed{\mathcal M_{fi}=\frac{1}{\sqrt2}G_Fg_{\mu\nu}\left(\bar\Psi_3\gamma^\mu(1-\gamma^5)\Psi_1\right)\left(\bar\Psi_4\gamma^\nu(1-\gamma^5)\Psi_2\right)}$ with $\boxed{\frac{G_F}{\sqrt2}=\frac{g_W^2}{8m_W^2}}$ because:

compare: $\mathcal M_{fi}=-\left(\frac{g_W}{\sqrt2}\bar\Psi_3\frac12\gamma^\mu(1-\gamma^5)\Psi_1\right)\frac{g_{\mu\nu}-\cancel{\frac{q_\mu q_\nu}{m_W^2}}}{\cancel{q^\alpha q_\alpha}-m_W^2}\left(\frac{g_W}{\sqrt2}\bar\Psi_4\frac12\gamma^\nu(1-\gamma^5)\Psi_2\right)\Longrightarrow$

$\mathcal M_{fi}=\frac{g_W^2}{8m_W^2}g_{\mu\nu}\left(\bar\Psi_3\gamma^\mu(1-\gamma^5)\Psi_1\right)\left(\bar\Psi_4\gamma^\nu(1-\gamma^5)\Psi_2\right)$

Strength of Weak Interaction

When $|q^\alpha q_\alpha|\ll m_w$ (low energy), then the propagator $P_W\sim\frac{1}{q^\alpha q_\alpha-m_W^2}$ becomes $P_W\sim-\frac{1}{m_W^2}$. In comparison, $P_{QWD}\sim\frac{1}{q^\alpha q_\alpha}$.

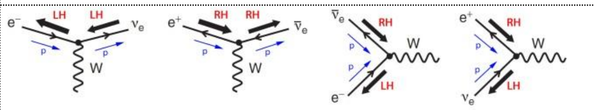
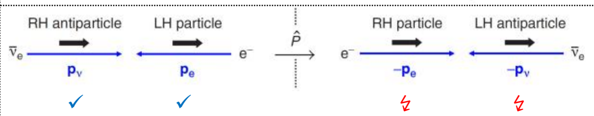
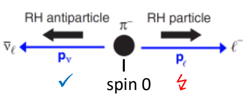
Therefore, weak ineraction decay rates, which are proprtional to $|\mathcal M|^2$, are suppressed by $\frac{(q^\alpha q_\alpha)^2}{m_W^4}$ realative to QED rates.

In the high energ limit, when $|q^\alpha q_\alpha|\gg m_w$, QED and weak interactions have almost the same strength.

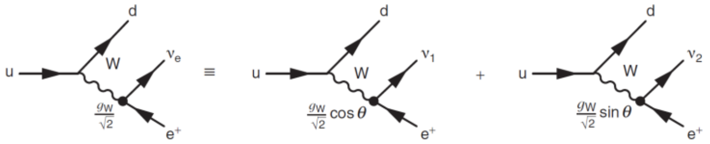
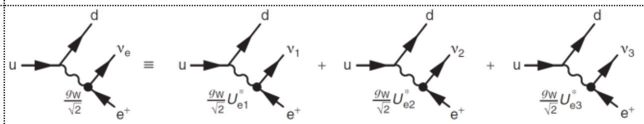
Decay Modes, Branching Ratio

General	Particles can have more than one possible final state or decay mode. Example: The K_s meson decays 99.9% of the time in one of two ways: $K_s \to \pi^+\pi^-$ and $K_s \to \pi^0\pi^0$.
Fermi's trans. rate and $\mathcal{M}$	Each decay mode has its own matrix element, $\mathcal{M}$. Fermi's Golden Rule gives us the transition rate Γ for each decay mode: $\Gamma(K_s \to \pi^+\pi^-) \propto \lvert\mathcal{M}(K_s \to \pi^+\pi^-)\rvert^2$ and $\Gamma(K_s \to \pi^0\pi^0) \propto \lvert\mathcal{M}(K_s \to \pi^0\pi^0)\rvert^2$
Tot. trans. rate	The total transition rate is equal to the sum of all allowed transition rates: $\Gamma(K_s) = \Gamma(K_s \to \pi^+\pi^-) + \Gamma(K_s \to \pi^0\pi^0)$
Branching ratio	The branching ratio, BR, is the fraction of time a particle decays to a particular final state $\mathrm{BR}(K_s \to \pi^+\pi^-) = \frac{\Gamma(K_s\to\pi^+\pi^-)}{\Gamma(K_s)}$

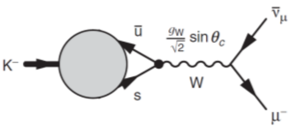
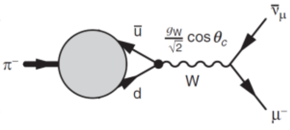
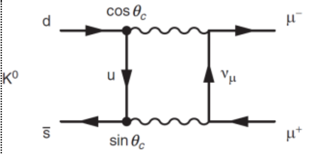
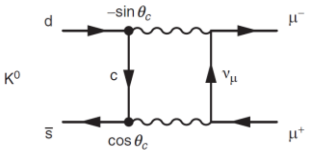
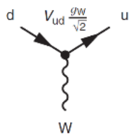
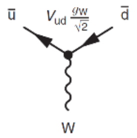
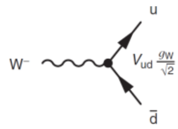
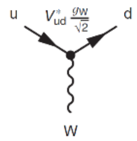
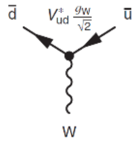
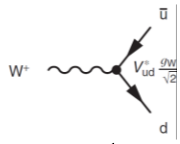
Chiral Structure of Weak Interaction

Chiral structure	The four-vector current is given by $j^\mu = \frac{g_W}{\sqrt{2}}\overline{U}(p^{\alpha\prime})\gamma^\mu \frac{1}{2}(1-\gamma^5)\,\overline{U}(p^\alpha) = \frac{g_W}{\sqrt{2}}\overline{U}(p^{\alpha\prime})\gamma^\mu \hat{P}_L\,\overline{U}(p^\alpha)$ Only **left-handed chiral particle states** and **right-handed chiral antiparticle states** participate in the charged current weak interaction.
Limit $E \gg m$	 These are the allowed helicity combinations in weak interaction vertices in the limit $E \gg m$ where helicity states are the same as chiral states. Only RH particles and LH antiparticles participate.
helicity ≈ chirality	 Under parity transformation $\hat{P}$ the $\vec{p}$ vectors are reversed, but the axial vector spins of the particles remain unchanged. Therefore the LH particle becomes RH, and the RH antiparticle becomes LH.
Chirality	In the realm of non-relativistic energies, the helicity must can be decomposed into RH and LH chiral components: $U_\uparrow = \frac{1}{2}\left(1+\frac{p}{E+m}\right)U_R + \frac{1}{2}\left(1-\frac{p}{E+m}\right)U_L$
Helicity in pion decay	Charged pions $\pi^\pm$ are $J^P = 0^-$ meson states formed from $u\bar{d}$ and $d\bar{u}$. They are the lightest mesons with $m_\pi \approx 140MeV$ and therefore cannot decay via the strong interaction; they can only decay through the weak interaction to states with lighter fundamental fermions. Hence $\pi^\pm$ pions can only decay to states with either electrons or muons. Main decay modes for π^-: (1) $\pi^- \to e^-\bar{\nu}_e$, (2) $\pi^- \to \mu^-\bar{\nu}_\mu$ (dominant) and (3) $\pi^- \to \mu^-\bar{\nu}_\mu\gamma$ The general expression for the decay rate $a \to 1+2$ is $\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2}\int\lvert\mathcal{M}_{fi}\rvert^2 d\Omega$ with $p^* = \frac{1}{2m_a}\sqrt{(m_a^2-(m_1+m_2)^2)(m_a^2-(m_1-m_2)^2)}$. Hence, we would expect the decay rate $\pi^- \to e^-\bar{\nu}_e$ to be greater than $\pi^- \to \mu^-\bar{\nu}_\mu$. The opposite is found to be true, charged pions decay almost entirely to $\pi^- \to \mu^-\bar{\nu}_\mu$ (or $\pi^+ \to \mu^+\nu_\mu$).
Qualitative explanation	 Neutrinos are effectively massless, $m_\nu \ll E \Longrightarrow$ for the neutrinos helicity=chirality. Therefore the antineutrino is always produced in a RH helicity=chirality state. Because the pion is spin 0, neutrino and lepton must have opposite spin. As they fly back-to-back, also the lepton has RH helicity. We can decompose RH helicity into RH and LH chirality. The electron has very little mass, therefore the chirality is almost completely RH (forbidden) with a very small LH part (allowed). The muon has much bigger mass, and therefore a significantly larger LH chirality part. The matrix element is proportional to the size of the LH chiral component: $\mathcal{M}_{l\pi} \sim \frac{1}{2}\left(1-\frac{p_l}{E_l+m_l}\right) \ldots (1)$ $\begin{pmatrix}E_\pi\\ \vec{p}_\pi\end{pmatrix} = \begin{pmatrix}E_\nu\\ \vec{p}_\nu\end{pmatrix} + \begin{pmatrix}E_l\\ \vec{p}_l\end{pmatrix}$ | in the π^- frame: $\vec{p}_\pi = 0, E_\pi = m_\pi$ $\Longrightarrow \begin{pmatrix}m_\pi\\ 0\end{pmatrix} = \begin{pmatrix}E_\nu\\ \vec{p}_\nu\end{pmatrix} + \begin{pmatrix}E_l\\ \vec{p}_l\end{pmatrix}$ | $E_\nu = \lvert\vec{p}_\nu\rvert \stackrel{\text{def}}{=} p_\nu \Longrightarrow$ $m_\pi = p_\nu + E_l \ldots (2)$, $0 = \vec{p}_\nu + \vec{p}_l \ldots (3)$ $(3) \Longrightarrow \vec{p}_\nu = -\vec{p}_l \Longrightarrow \lvert\vec{p}_\nu\rvert = \lvert\vec{p}_l\rvert \Longrightarrow p_\nu = p_l \stackrel{(2)}{\Longrightarrow} m_\pi = p_l + E_l \Longrightarrow E_l = m_\pi - p_l \ldots (4)$ $E_l^2 = m_\pi^2 + p_l^2 - 2m_\pi p_l \lvert E_l^2 = m_l^2 + p_l^2 \Longrightarrow m_l^2 + \cancel{p_l^2} = m_\pi^2 + \cancel{p_l^2} - 2m_\pi p_l \Longrightarrow 2m_\pi p_l = m_\pi^2 - m_l^2 \Longrightarrow p_l = \frac{m_\pi^2-m_l^2}{2m_\pi} \ldots (5) \stackrel{(4)}{\Longrightarrow}$ $E_l = m_\pi - \frac{m_\pi^2-m_l^2}{2m_\pi} = \frac{2m_\pi^2-m_\pi^2+m_l^2}{2m_\pi} \Longrightarrow E_l = \frac{m_\pi^2+m_l^2}{2m_\pi} \ldots (6)$ $\quad (1) \Longrightarrow \mathcal{M}_{l\pi} \sim \frac{1}{2}\left(1-\frac{p_l}{E_l+m_l}\right) \stackrel{(5),(6)}{\Longrightarrow} \mathcal{M}_{l\pi} \sim \frac{1}{2}\left(1-\frac{m_\pi^2-m_l^2}{2m_\pi}\frac{1}{\frac{m_\pi^2+m_l^2}{2m_\pi}+m_l}\right) \Longrightarrow$ $\mathcal{M}_{l\pi} \sim \frac{1}{2}\left(1-\frac{m_\pi^2-m_l^2}{\cancel{2m_\pi}}\frac{\cancel{2m_\pi}}{m_\pi^2+m_l^2+2m_\pi m_l}\right) = \frac{1}{2}\left(1-\frac{(m_\pi-m_l)(m_\pi+m_l)}{(m_\pi+m_l)^2}\right) = \frac{1}{2}\left(1-\frac{m_\pi-m_l}{m_\pi+m_l}\right) = \frac{1}{2}\left(\frac{m_\pi+m_l-m_\pi+m_l}{m_\pi+m_l}\right) = \frac{1}{2}\frac{2m_l}{m_\pi+m_l} \Longrightarrow \boxed{\mathcal{M}_{l\pi} \sim \frac{m_l}{m_\pi+m_l}}$
Probability	Probability for emission under wrong helicity (RH particle or LH antiparticle): $\boxed{p_{RH}^{\text{particle}} = p_{LH}^{\text{antiparticle}} = \frac{1}{2}(1-\beta)}$

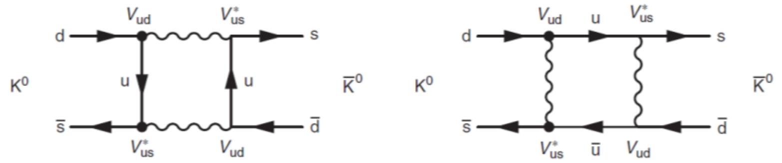
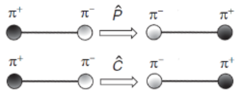
Neutrino Oscillation, Mass and Weak Neutrino Eigenstates, PMNS Matrix

General: weak and mass eigen-states	Neutrinos created in weak interactions ($\boldsymbol{\nu_e, \nu_\mu, \nu_\tau}$) are called "**weak eigenstates**" or "**flavor eigenstates**". They are a superposition of more fundamental "**mass eigenstates**" $\boldsymbol{\nu_1, \nu_2, \nu_3}$ which represent three fundamental neutrinos with small and slightly (unknown) different masses. Weak-force couplings compel the simultaneously emitted neutrino to be in a "charged-lepton-centric" superposition such as, for example, ν_e, which is an eigenstate for a flavor that is fixed by the electron's mass eigenstate, and not in one of the ν_1, ν_2, ν_3 neutrino's own mass eigenstates.
Neutrino oscillation	Neutrino oscillation arises from mixing between the flavor eigenstates ν_e, ν_μ, ν_τ and mass eigenstates ν_1, ν_2, ν_3. That is, the flavor eigenstates ν_e, ν_μ, ν_τ that interact with the charged leptons in weak interactions are each a different superposition of the three (propagating) mass eigenstates ν_1, ν_2, ν_3 of definite mass. Neutrinos are emitted and absorbed in weak processes in flavor eigenstates ν_e, ν_μ, ν_τ, but travel as mass eigenstates ν_1, ν_2, ν_3. As a neutrino superposition propagates through space, the quantum mechanical phases of the three neutrino mass eigenstates ν_1, ν_2, ν_3 advance at slightly different rates, due to the slight differences in their respective masses. This results in a changing superposition mixture of mass eigenstates ν_e, ν_μ, ν_τ as the neutrino travels; but a different mixture of mass eigenstates ν_1, ν_2, ν_3 corresponds to a different mixture of flavor states ν_e, ν_μ, ν_τ. So a neutrino born as, say, an electron neutrino ν_e will be some mixture of ν_e, ν_μ, ν_τ after traveling some distance. Since the quantum mechanical phase advances in a periodic fashion, after some distance the state will nearly return to the original mixture, and the neutrino will be again mostly electron neutrino ν_e. The electron flavor content of the neutrino will then continue to oscillate – as long as the quantum mechanical state maintains coherence. Since mass differences between neutrino flavors are small in comparison with long coherence lengths for neutrino oscillations, this microscopic quantum effect becomes observable over macroscopic distances.
Oscillation of two flavors (simplified example)	The main feature can be understood by considering just two flavors. We consider the flavor ("weak") eigenstates ν_e, ν_μ, which here are taken to be coherent superpositions of the mass eigenstates ν_1, ν_2. The mass eigenstates propagate as follows:  $\lvert\nu_1(\vec{x},t)\rangle = \lvert\nu_1\rangle e^{i(\vec{p}_1\cdot\vec{x}-E_1 t)} = \lvert\nu_1\rangle e^{-ip_1^\mu x_\mu^1}$ $\lvert\nu_2(\vec{x},t)\rangle = \lvert\nu_2\rangle e^{i(\vec{p}_2\cdot\vec{x}-E_2 t)} = \lvert\nu_2\rangle e^{-ip_2^\mu x_\mu^2}$ In this simplified two-flavor example the flavor eigenstates ν_e, ν_μ are related to the mass eigenstates ν_1, ν_2 by a 2x2 unitary matrix that can be expressed in terms of a single mixing angle ϑ: $\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2}\\ U_{\mu 1} & U_{\mu 2}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos(\vartheta) & \sin(\vartheta)\\ -\sin(\vartheta) & \cos(\vartheta)\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$ ... (1) The wavefunction of the neutrino at time $t=0$: $\lvert\Psi(0)\rangle = \lvert\nu_e\rangle = \cos(\vartheta)\lvert\nu_1\rangle + \sin(\vartheta)\lvert\nu_2\rangle$ ... (2) The state subsequently evolves according to the tie dependence of the mass eigenstates: $\lvert\Psi(\vec{x},t)\rangle = \cos(\vartheta)\lvert\nu_1\rangle e^{-ip_1^\mu x_\mu^1} + \sin(\vartheta)\lvert\nu_2\rangle e^{-ip_2^\mu x_\mu^2} = \cos(\vartheta)\lvert\nu_1\rangle e^{-i\phi_1} + \sin(\vartheta)\lvert\nu_2\rangle e^{-i\phi_2}$ ... (3) with $\phi_i = E_i t - \vec{p}_i\cdot\vec{x}$ By inverting (1) we get: $\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos(\vartheta) & -\sin(\vartheta)\\ \sin(\vartheta) & \cos(\vartheta)\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} \Rightarrow$ $\lvert\nu_1\rangle = \cos(\vartheta)\lvert\nu_e\rangle - \sin(\vartheta)\lvert\nu_\mu\rangle$, $\lvert\nu_2\rangle = \sin(\vartheta)\lvert\nu_e\rangle + \cos(\vartheta)\lvert\nu_\mu\rangle$ $\overset{(3)}{\Rightarrow}$ $\lvert\Psi(\vec{x},t)\rangle = \cos(\vartheta)\big(\cos(\vartheta)\lvert\nu_e\rangle - \sin(\vartheta)\lvert\nu_\mu\rangle\big)e^{-i\phi_1} + \sin(\vartheta)\big(\sin(\vartheta)\lvert\nu_e\rangle + \cos(\vartheta)\lvert\nu_\mu\rangle\big)e^{-i\phi_2}$ $\lvert\Psi(\vec{x},t)\rangle = e^{-i\phi_1}\cos^2(\vartheta)\lvert\nu_e\rangle - e^{-i\phi_1}\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle + e^{-i\phi_2}\sin^2(\vartheta)\lvert\nu_e\rangle + e^{-i\phi_2}\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle$ $\lvert\Psi(\vec{x},t)\rangle = (e^{-i\phi_1}\cos^2(\vartheta) + e^{-i\phi_2}\sin^2(\vartheta))\lvert\nu_e\rangle - (e^{-i\phi_1} - e^{-i\phi_2})\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle$ $\lvert\Psi(\vec{x},t)\rangle = e^{-i\phi_1}(\cos^2(\vartheta) + e^{i\phi_1}e^{-i\phi_2}\sin^2(\vartheta))\lvert\nu_e\rangle - e^{-i\phi_1}(1 - e^{i\phi_1}e^{-i\phi_2})\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle$ $\lvert\Psi(\vec{x},t)\rangle = e^{-i\phi_1}\Big(\big(\cos^2(\vartheta) + e^{i(\phi_1-\phi_2)}\sin^2(\vartheta)\big)\lvert\nu_e\rangle - \big(1 - e^{i(\phi_1-\phi_2)}\big)\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle\Big)$ $\boxed{\lvert\Psi(\vec{x},t)\rangle = e^{-i\phi_1}\Big(\big(\cos^2(\vartheta) + e^{i\Delta\phi}\sin^2(\vartheta)\big)\lvert\nu_e\rangle - \big(1 - e^{i\Delta\phi}\big)\cos(\vartheta)\sin(\vartheta)\lvert\nu_\mu\rangle\Big) \text{ with } \Delta\phi = \phi_1 - \phi_2}$
Oscillation of three flavors: PMNS matrix	 $\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$ ... unitary $U^{-1} = U^\dagger = (U^*)^T \Rightarrow \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^*\\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^*\\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^*\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$ magni-tudes: $\begin{pmatrix}\lvert U_{e1}\rvert & \lvert U_{e2}\rvert & \lvert U_{e3}\rvert\\ \lvert U_{\mu 1}\rvert & \lvert U_{\mu 2}\rvert & \lvert U_{\mu 3}\rvert\\ \lvert U_{\tau 1}\rvert & \lvert U_{\tau 2}\rvert & \lvert U_{\tau 3}\rvert\end{pmatrix} = \begin{pmatrix}0.85 & 0.50 & 0.17\\ 0.35 & 0.60 & 0.70\\ 0.35 & 0.60 & 0.70\end{pmatrix}$
repr. with 3 Euler angles and phase	$U_{PMNS} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\vartheta_{23}) & \sin(\vartheta_{23})\\ 0 & -\sin(\vartheta_{23}) & \cos(\vartheta_{23})\end{pmatrix}\begin{pmatrix}\cos(\vartheta_{13}) & 0 & \sin(\vartheta_{13})\,e^{-i\delta}\\ 0 & 1 & 0\\ -\sin(\vartheta_{13})\,e^{-i\delta} & 0 & \cos(\vartheta_{13})\end{pmatrix}\begin{pmatrix}\cos(\vartheta_{12}) & \sin(\vartheta_{12}) & 0\\ -\sin(\vartheta_{12}) & \cos(\vartheta_{12}) & 0\\ 0 & 0 & 1\end{pmatrix}$ $\vartheta_{12} = 35°$ $\vartheta_{23} = 45°$ $\vartheta_{13} = 10°$

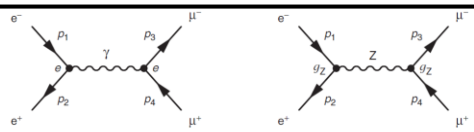
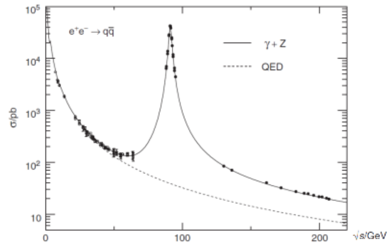
Weak Interaction of Quarks, Quark Mixing, Cabibbo Angle and CKM Matrix

General:	There is an universal coupling strength of the weak interaction $W^\pm$ to charged leptons ($e^\pm, \mu^\pm, \tau^\pm$) and the corresponding neutrino weak (flavor) eigenstates (ν_e, ν_μ, ν_τ): $G_F^{(e)} = G_F^{(\mu)} = G_F^{(\tau)}$. However, the coupling strength $G_F^{(\beta)}$ at the ud quark weak interaction vertex is found to be 5% smaller. Furthermore, different coupling strengths are e.g. found for the ud and us weak charged current vertices.
Cabibbo hypothesis:	In the Cabibbo hypothesis, the weak interaction of quarks have the same strengths as the leptons, but the object that couples e.g. to the u quark via charged-current weak interaction is not just the d quark, but rather a superposition of the down-type quarks d and s, here denoted by $\vert d'\rangle = V_{ud}\vert d\rangle + V_{us}\vert s\rangle$. $\vert V_{ud}\vert^2$ and $\vert V_{us}\vert^2$ are the probabilities that d and s decay into u quarks. $\begin{matrix} u & & c & & u \\ \vert V_{ud}\vert^2 \uparrow & \nwarrow & \vert V_{us}\vert^2 & \Longrightarrow G_F & \uparrow \\ d & & s & & d' \end{matrix}$ Similarly, the object that couples to the c quark via charged-current weak interaction is not just the s quark, but rather another superposition of the down-type quarks d and s, here denoted by $\vert s'\rangle = V_{cd}\vert d\rangle + V_{cs}\vert s\rangle$. $\vert V_{cd}\vert^2$ and $\vert V_{cs}\vert^2$ are the probabilities that d and s decay into c quarks. $\begin{matrix} u & & c & & c \\ \vert V_{cd}\vert^2 & \nearrow & \uparrow \vert V_{cs}\vert^2 & \Longrightarrow G_F & \uparrow \\ d & & s & & s' \end{matrix}$ The weak (flavor) eigenstates d′ and s′ are related to the mass eigenstates d and s by a 2x2 unitary matrix that can be expressed in terms of the single Cabbibo mixing angle $\vartheta_c = 13.02°$. $\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \cos(\vartheta_c) & \sin(\vartheta_c) \\ -\sin(\vartheta_c) & \cos(\vartheta_c) \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix}$
Main decay modes of π^- and K^-	$\left.\begin{matrix} \Gamma(K^- \to \mu^-\bar{\nu}_\mu) \propto \sin^2(\vartheta_c) \\ \Gamma(\pi^- \to \mu^-\bar{\nu}_\mu) \propto \cos^2(\vartheta_c) \end{matrix}\right\} \Longrightarrow$ The K^- decay rate is suppressed by a factor of $\frac{\Gamma(K^- \to \mu^-\bar{\nu}_\mu)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} = \frac{\sin^2(\vartheta_c)}{\cos^2(\vartheta_c)} = \tan^2(\vartheta_c)$
Flavor changing neutral-current decay	The flavor changing neutral-current decay of neutral mesons can be represented by box-diagrams showing the exchange of virtual quarks. For example, the decay of the neutral kaon $K_L^0 \to \mu^+\mu^-$ can occur via the exchange of a virtual up-quark, and also via the exchange of a virtual charm-quark: $\mathcal{M}_u \propto g_w^4 \cos(\vartheta_c)\sin(\vartheta_c)$ $\mathcal{M}_c \propto -g_w^4 \cos(\vartheta_c)\sin(\vartheta_c)$ Because both diagrams give the same final state: $\vert\mathcal{M}\vert^2 = \vert\mathcal{M}_u + \mathcal{M}_c\vert^2 \approx 0$ The cancellation is not exact because $m_c \neq m_u$ and hence this explains the small branching ratio.
CKM Matrix	$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}\begin{pmatrix} d \\ s \\ b \end{pmatrix}$ with d' coupling to u, s' coupling to c, and b' coupling to t Magnitudes: $\begin{pmatrix} \vert V_{ud}\vert & \vert V_{us}\vert & \vert V_{ub}\vert \\ \vert V_{cd}\vert & \vert V_{cs}\vert & \vert V_{cb}\vert \\ \vert V_{td}\vert & \vert V_{ts}\vert & \vert V_{tb}\vert \end{pmatrix} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$
repr. with 3 Euler angles and phase	$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\vartheta_{23}) & \sin(\vartheta_{23}) \\ 0 & -\sin(\vartheta_{23}) & \cos(\vartheta_{23}) \end{pmatrix}\begin{pmatrix} \cos(\vartheta_{13}) & 0 & \sin(\vartheta_{13})\,e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin(\vartheta_{13})\,e^{-i\delta} & 0 & \cos(\vartheta_{13}) \end{pmatrix}\begin{pmatrix} \cos(\vartheta_{12}) & \sin(\vartheta_{12}) & 0 \\ -\sin(\vartheta_{12}) & \cos(\vartheta_{12}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\vartheta_{12} = 13°$, $\vartheta_{23} = 2.3°$, $\vartheta_{13} = 0.2°$
Wolfenstein parametrizetion	$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$ Charged current vertices: $-i\frac{g_W}{\sqrt{2}}(\bar{u},\bar{c},\bar{t})\gamma^\mu\frac{1}{2}(1-\gamma^5)\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$
Charged current vertices	The vertex factor contains V_{ud} (or any other matching V_{ij}) when the down-type (charge $-\frac{1}{3}$) particle's arrow points inwards $-i\frac{g_W}{\sqrt{2}}V_{ud}\,\bar{\mathrm{U}}(u)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{U}(d)$ $-i\frac{g_W}{\sqrt{2}}V_{ud}\,\bar{\mathrm{V}}(\bar{u})\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{V}(\bar{d})$ $-i\frac{g_W}{\sqrt{2}}V_{ud}\,\bar{\mathrm{U}}(u)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{V}(\bar{d})$ The vertex factor contains V_{ud}^* (or any other matching V_{ij}^*) when the down-type (charge $-\frac{1}{3}$) particle's arrow points outwards $-i\frac{g_W}{\sqrt{2}}V_{ud}^*\,\bar{\mathrm{U}}(d)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{U}(u)$ $-i\frac{g_W}{\sqrt{2}}V_{ud}^*\,\bar{\mathrm{V}}(\bar{d})\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{V}(\bar{u})$ $-i\frac{g_W}{\sqrt{2}}V_{ud}^*\,\bar{\mathrm{U}}(d)\,\gamma^\mu\frac{1}{2}(1-\gamma^5)\,\mathrm{V}(\bar{u})$

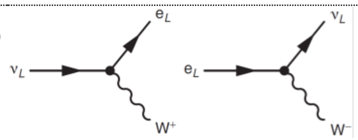
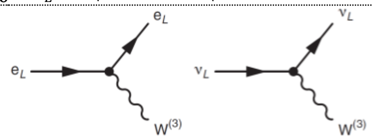
The Neutral Kaon System

General:	The weak interaction provides a mechanism whereby the neutral kaons $\mathrm{K}^0(d\bar{s})$ and $\overline{\mathrm{K}}^0(s\bar{d})$ can mix (see 2 of the 9 possible diagrams on the right side) 
K-short K_S and K-long K_L	Beause of the $\mathrm{K}^0 \leftrightarrow \overline{\mathrm{K}}^0$ mixing, a neutral kaon produced as a K^0 will develop a $\overline{\mathrm{K}}^0$ component (K^0 and $\overline{\mathrm{K}}^0$ are flavour eigenstates). The $\mathrm{K}^0/\overline{\mathrm{K}}^0$ system has to be considered as whole. Neutral kaons propagate as linear combinations of the K^0 and $\overline{\mathrm{K}}^0$. These physical states are known as the "K-short" K_S and the "K-long" K_L kaon.
Spin-parity and $\hat{C}\hat{P}$	K^0 and $\overline{\mathrm{K}}^0$ have spin-parity $J^P = 0^- \Longrightarrow$ $\hat{P}\|K^0\rangle = -\|K^0\rangle \quad \hat{P}\|\bar{K}^0\rangle = -\|\bar{K}^0\rangle$ by convection: $\hat{C}\|K^0\rangle = -\|\bar{K}^0\rangle \quad \hat{C}\|\bar{K}^0\rangle = -\|K^0\rangle$ $\Longrightarrow$ $\hat{C}\hat{P}\|K^0\rangle = +\|\bar{K}^0\rangle \quad \hat{C}\hat{P}\|\bar{K}^0\rangle = +\|K^0\rangle$
$\|K_S\rangle, \|K_L\rangle$	$\|K_1^0\rangle = \frac{1}{\sqrt{2}}(\|K^0\rangle + \|\bar{K}^0\rangle)$ is a $\hat{C}\hat{P}$ eigenstate with $\hat{C}\hat{P}\|K_1^0\rangle = \frac{1}{\sqrt{2}}\hat{C}\hat{P}(\|\bar{K}^0\rangle + \|K^0\rangle) = \|K_1^0\rangle \Longrightarrow$ $\mathrm{CP}(\|K_1^0\rangle) = +1$ ... $(1a)$ $\|K_2^0\rangle = \frac{1}{\sqrt{2}}(\|K^0\rangle - \|\bar{K}^0\rangle)$ is a $\hat{C}\hat{P}$ eigenstate with $\hat{C}\hat{P}\|K_2^0\rangle = \frac{1}{\sqrt{2}}\hat{C}\hat{P}(\|\bar{K}^0\rangle - \|K^0\rangle) = -\|K_2^0\rangle \Longrightarrow$ $\mathrm{CP}(\|K_2^0\rangle) = -1$ ... $(1b)$ If $\hat{C}\hat{P}$ were conserved in the weak interaction, these states would be the $\|K_S\rangle$ and $\|K_L\rangle$ states. But $\hat{C}\hat{P}$ violation is small, hence, in good approximation $\|K_S\rangle \approx \|K_1^0\rangle = \frac{1}{\sqrt{2}}(\|K^0\rangle + \|\bar{K}^0\rangle),\ \|K_L\rangle \approx \|K_2^0\rangle = \frac{1}{\sqrt{2}}(\|K^0\rangle - \|\bar{K}^0\rangle)$ $\tau_S = 900ps, \tau_L = 500ns$
Kaon decay to pions	$\Gamma(K_S \to \pi\pi) \gg \Gamma(K_S \to \pi\pi\pi)$ and $\Gamma(K_L \to \pi\pi\pi) \gg \Gamma(K_L \to \pi\pi)$ **(K_S decay mostly to $\pi\pi$, and K_L decay mostly to $\pi\pi\pi$)** **Explanation:** Because kaons and pions have $J^P = 0^- \Longrightarrow \mathrm{P}(\pi^0\pi^0) = (-1)^l\,\mathrm{P}(\pi^0)\,\mathrm{P}(\pi^0) = (-1)^0(-1)(-1) = +1$ ... $(2a)$ $\hat{C}\|\pi^0\rangle = \frac{1}{\sqrt{2}}\hat{C}(u\bar{u} - d\bar{d}) = \frac{1}{\sqrt{2}}\hat{C}(\bar{u}u - \bar{d}d) = +\|\pi^0\rangle \Longrightarrow \mathrm{C}(\pi^0\pi^0) = \mathrm{C}(\pi^0)\,\mathrm{C}(\pi^0) = (+1)(+1) = +1$ ... $(2b)$ $\mathrm{CP}(\pi^0\pi^0) = \mathrm{C}(\pi^0\pi^0)\,\mathrm{P}(\pi^0\pi^0) \overset{(2ab)}{=} (+1)(+1) \Longrightarrow$ $\mathrm{CP}(\pi^0\pi^0) = +1$ ... (3) $\mathrm{P}(\pi^+\pi^-) = (-1)^l\,\mathrm{P}(\pi^-)\,\mathrm{P}(\pi^-) = (-1)^0(-1)(-1) = +1$ ... $(4a)$ $\hat{C}$ and $\hat{P}$ have the same effect on $\pi^+\pi^-$ (see image): $\mathrm{C}(\pi^+\pi^-) = \mathrm{P}(\pi^+\pi^-) \overset{(4a)}{=} +1$ ... $(4b)$ $\mathrm{CP}(\pi^+\pi^-) = \mathrm{C}(\pi^+\pi^-)\,\mathrm{P}(\pi^+\pi^-) \overset{(4ab)}{=} (+1)(+1) \Longrightarrow$ $\mathrm{CP}(\pi^+\pi^-) = +1$ ... (5)  With some more involved derivations: $\mathrm{CP}(\pi^0\pi^0\pi^0) = -1$ ... (6) $\mathrm{CP}(\pi^+\pi^-\pi^0) = -1$ ... (7) **If $\widehat{C}\widehat{P}$ were conserved in the weak interaction, the hadronic decay of the $\widehat{C}\widehat{P}$ eigenstates $\|K_1^0\rangle$ and $\|K_2^0\rangle$ would be exclusively $K_1^0 \to \pi\pi$ and $K_2^0 \to \pi\pi\pi$, because $\mathrm{CP}(\|K_1^0\rangle) = +1$ and $\mathrm{CP}(\pi\pi) = +1$, and $\mathrm{CP}(\|K_2^0\rangle) = -1$ and $\mathrm{CP}(\pi\pi\pi) = -1$.**
CP violation in hadronic kaon decays	If a neutral kaon is produced in the strong interaction $p\bar{p} \to K^-\pi^+K^0$, at the tim eof production, the kaon is in the flavor eigenstate: $\|\mathrm{K}(0)\rangle = \|K^0\rangle$. Without $\widehat{\mathrm{CP}}$ violation where $\|K_S\rangle = \|K_1^0\rangle$ and $\|K_L\rangle = \|K_2^0\rangle$ the flavor state can be written n terms of the $\widehat{\mathrm{CP}}$ eigenstates: $\|\mathrm{K}(0)\rangle = \|K^0\rangle = \frac{1}{\sqrt{2}}(\|K_1^0\rangle + \|K_2^0\rangle) = \frac{1}{\sqrt{2}}(\|K_S\rangle + \|K_L\rangle)$ Proof: $\|K^0\rangle = \frac{1}{\sqrt{2}}(\|K_1^0\rangle + \|K_2^0\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\|K^0\rangle + \|\bar{K}^0\rangle) + \frac{1}{\sqrt{2}}(\|K^0\rangle - \|\bar{K}^0\rangle)\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\|K^0\rangle + \frac{1}{\sqrt{2}}\|\bar{K}^0\rangle + \frac{1}{\sqrt{2}}\|K^0\rangle - \frac{1}{\sqrt{2}}\|\bar{K}^0\rangle\right)$ $\|K^0\rangle = \frac{1}{2}\|K^0\rangle + \frac{1}{2}\|K^0\rangle$ ✓ The subsequent time evolution is described in terms of the K_S and K_L, which are the observed kaons In the rest fram of the kaon: $\|\mathrm{K}_S(t)\rangle = \|\mathrm{K}_S(t)\rangle e^{-imst}e^{-t/\tau_S}$ and $\|\mathrm{K}_L(t)\rangle = \|\mathrm{K}_L(t)\rangle e^{-im_Lt}e^{-t/\tau_L} \Longrightarrow$ If a kaon beam, which originally consisted of K^0 propagates over a large distance $L \gg c\tau_S$, the K_S component will decay away, leaving a pure K_L beam. **Therefore, if $\widehat{C}\widehat{P}$ was conserved, at large distances from the production of the beam, the decays to two pions would never be detected. But, even at a large distance, some $K_L \to \pi\pi$ decays are observed $\Longrightarrow$ CP violation!**
Origins of CP violation	**(1) $\widehat{\mathrm{CP}}$ violation in $K^0 \leftrightarrow \bar{K}^0$ mixing (main contribution):** $\|K_S\rangle = \frac{1}{\sqrt{1+\|\varepsilon\|^2}}(\underbrace{\|K_1^0\rangle}_{\pi\pi} + \varepsilon\underbrace{\|K_2^0\rangle}_{\pi\pi\pi})$ and $\|K_L\rangle = \frac{1}{\sqrt{1+\|\varepsilon\|^2}}(\underbrace{\|K_2^0\rangle}_{\pi\pi\pi} + \varepsilon\underbrace{\|K_1^0\rangle}_{\pi\pi})$ **(2) Direct $\widehat{\mathrm{CP}}$ violation in the decay of a $\widehat{\mathrm{CP}}$ eigenstate:** $\|K_L\rangle = \underbrace{\|K_2^0\rangle}_{\pi\pi\pi,\ \pi\pi}$ **(negligible)**

Z Resonance

General:	Because the Z boson couples to all flavors of fermions, the photon in any QED process can be replaced by a Z boson. The respective coupling terms in the matrix elements are $\mathcal{M}_\gamma \propto \frac{e^2}{q^\mu q_\mu}$ and $\mathcal{M}_Z \propto \frac{g_Z^2}{q^\mu q_\mu - m_Z^2}$ 
Z resonance	• for low center-of-mass energy $\sqrt{q^\mu q_\mu} = \sqrt{s} \ll m_z$ QED process dominates • for high center-of-mass energy $\sqrt{q^\mu q_\mu} = \sqrt{s} \gg m_z$ QED and Z exchange processes are both important because coupling strength of γ and Z are comparable • in the region $\sqrt{q^\mu q_\mu} = \sqrt{s} \approx m_z$ Z boson process dominates (Z resonance) 
Avoiding divergence at $q^\mu q_\mu = m_z^2$	Wavefunction of unstable Z boson with total decay rate $\Gamma = \frac{1}{\tau}$: $\Psi \propto e^{-im_zt}e^{-\frac{\Gamma}{2}t}$ so that $\Psi^*\Psi \propto e^{-\Gamma t}$ This can be introduced to the free particle form $\Psi \propto e^{-im_zt}$ by substituting $m_z \to m_z - i\frac{\Gamma_z}{2}$. Making the same replacement in the Z-boson propagator: $m_z^2 \to \left(m_z - i\frac{\Gamma_z}{2}\right)^2 = m_z^2 - im_z\Gamma_z - \frac{1}{4}\Gamma_z^2 \approx m_z^2 - im_z\Gamma_z \Longrightarrow \frac{g_Z^2}{q^\mu q_\mu - m_z^2} \to \frac{g_Z^2}{q^\mu q_\mu - m_z^2 + im_Z\Gamma_Z}$

The Weak interaction SU(2)$_L$ Group

SU(2) phase transformation	The charged-current weak interaction is associated with invariance under SU(2) local phase transformations: $\varphi(x) \rightarrow \varphi'(x) = e^{ig_W \vec{\alpha}(x)\cdot\vec{T}}\,\varphi(x)$ … (1) with $\vec{T}$ containing the three generators of SU(2): $\vec{T} = \frac{1}{2}(\sigma_1, \sigma_2, \sigma_3)^T$
Gauge fields	The required local gauge invariance can only be satisfied by the introduction of three gauge fields W_μ^k with $k = 1,2,3$ corresponding to three gauge bosons $W^{(1)}, W^{(2)}, W^{(3)}$
weak isospin doublets	Because the generators of the SU(2) are the Pauli matrices, $\varphi(x)$ must be written in terms of two components and is termed "weak isospin doublet". Since the $W^\pm$ couples together fermions differing by one unit charge, the weak isospin doublet must contain flavors differing by one unit of electric charge. Since the weak charged interaction only couples to LH particles and RH antiparticles, these doublets only contain LH particles and RH antiparticle chiral states. RH particle and LH antiparticle chiral states are placed in weak isospin singlets. The weak isospin doublets are constructed from the weak eigenstates and therefore account for the mixing in the CKM and PMNS matrices. The upper member of the doublet is always the particle with differs by plus one unit electric charge relative to the lower member and are weak isospin $I_W = \frac{1}{2}$ states. e.g. $\varphi(x) = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L$ 3rd component of weak isospin $\hat{I}_W^{(3)}\nu_e = +\frac{1}{2}\nu_e$ and $\hat{I}_W^{(3)}e^- = -\frac{1}{2}e^-$ all doublets $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L, \begin{pmatrix} t \\ b' \end{pmatrix}_L$
weak isospin singlets	$e_R^-, \mu_R^-, \tau_R^-, u_R, c_R, t_R, d_R, s_R, b_R$ with weak Isospin $I_W = 0$ and third component of weak isospin $I_W^{(3)} = 0$
Interact. term	$ig_W \hat{T}_k \gamma^\mu W_\mu^k \varphi_L = ig_W \frac{1}{2}\hat{\sigma}_k \gamma^\mu W_\mu^k \varphi_L$ weak currents e.g. for $\varphi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \Longrightarrow j_1^\mu = \frac{g_W}{2}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_1 \varphi_L \quad j_2^\mu = \frac{g_W}{2}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_2 \varphi_L \quad j_3^\mu = \frac{g_W}{2}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_3 \varphi_L$
$W^\pm$ currents $W^\pm$ bosons	The weak charged currents are related to the ladder operators $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_1 \pm i\hat{\sigma}_2)$ The $W^\pm$ currents are: $j_\pm^\mu = \frac{1}{\sqrt{2}}(j_1^\mu \pm i j_2^\mu)$ The physical $W^\pm$ bosons are: $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^{(1)} \pm i W_\mu^{(2)})$  All weak currents: $\vec{j}^\mu \cdot \vec{W}_\mu = j_1^\mu W_\mu^{(1)} + j_2^\mu W_\mu^{(2)} + j_3^\mu W_\mu^{(3)} \equiv j_+^\mu W_\mu^+ + j_-^\mu W_\mu^- + j_3^\mu W_\mu^{(3)}$ W^+ exchange: $j_+^\mu = \frac{g_W}{\sqrt{2}}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_+ \varphi_L = \frac{g_W}{\sqrt{2}}\overline{\varphi}_L \gamma^\mu \frac{1}{2}(\hat{\sigma}_1 + i\hat{\sigma}_2)\varphi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L\ \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}}\overline{\nu}_L \gamma^\mu e_L = \frac{g_W}{\sqrt{2}}\overline{\nu}\gamma^\mu \frac{1}{2}(1-\gamma^5)e$ W^- exchange: $j_-^\mu = \frac{g_W}{\sqrt{2}}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_- \varphi_L = \frac{g_W}{\sqrt{2}}\overline{\varphi}_L \gamma^\mu \frac{1}{2}(\hat{\sigma}_1 - i\hat{\sigma}_2)\varphi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L\ \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \frac{g_W}{\sqrt{2}}\overline{e}_L \gamma^\mu \nu_L = \frac{g_W}{\sqrt{2}}\overline{e}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu$
Weak neutral current	$j_3^\mu = \frac{g_W}{2}\overline{\varphi}_L \gamma^\mu \hat{\sigma}_3 \varphi_L = \frac{g_W}{2}(\overline{\nu}_L\ \overline{e}_L)\gamma^\mu \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = g_W \frac{1}{2}\overline{\nu}_L \gamma^\mu \nu_L - g_W \frac{1}{2}\overline{e}_L \gamma^\mu e_L$ $j_3^\mu = \hat{I}_W^{(3)} g_W \overline{f}\gamma^\mu \frac{1}{2}(1-\gamma^5)f$ 

Lagrangians in QFT

Classical discrete particles	Lagrangian: $L=T-V$ with $T=E_{kin}, V=E_{pot}$ | Euler-Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)-\frac{\partial L}{\partial q_i}=0$ | Example: Particle moving in 1D: $L=T-V=\frac{mv^2}{2}-V(x)=\frac{m\dot x^2}{2}-V(x)$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right)-\frac{\partial L}{\partial x}=\frac{d}{dt}(m\dot x)-\left(\frac{\partial V}{\partial x}\right)=m\ddot x+\frac{\partial V}{\partial x}=0\Longrightarrow m\ddot x=-\frac{\partial V}{\partial x}=F$
scalar fields	$q_i\to\phi_i(t,x,y,z)$ $\mathrm{L}(q_i,\dot q_i)\to\mathcal{L}(\phi_i,\partial_\mu\phi_i)$ $\dot q_i\to\partial_\mu\phi_i$ with $\mathrm{L}=\int\mathcal{L}\,d^3x$ | Euler-Lagrange $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)-\frac{\partial\mathcal{L}}{\partial\phi_i}=0$ ... (1) with ϕ_i ... scalar field
relativistic (spin 0) scalar field	Free non-interacting scalar field $\mathcal{L}_S=\frac12\partial_\mu\phi\partial^\mu\phi-\frac12m^2\phi^2$ ... (2) $\frac{\partial\mathcal{L}_S}{\partial(\partial_\mu\phi_i)}\overset{(2)}{=}\frac{\partial}{\partial(\partial_\mu\phi_i)}\left(\frac12\partial_\mu\phi\partial^\mu\phi-\frac12m^2\phi^2\right)=\frac12\left(\frac{\partial}{\partial(\partial_\mu\phi_i)}\partial_\mu\phi\right)\partial^\mu\phi+\frac12\partial_\mu^{\uparrow}\phi\left(\frac{\partial}{\partial(\partial_\mu\phi_i)}\partial^\mu_{\downarrow}\phi\right)$ $\frac{\partial\mathcal{L}_S}{\partial(\partial_\mu\phi_i)}=\frac12\partial^\mu\phi+\frac12\partial^\mu\phi\left(\frac{\partial}{\partial(\partial_\mu\phi_i)}\partial_\mu\phi\right)=\frac12\partial^\mu\phi+\frac12\partial^\mu\phi=\partial^\mu\phi\overset{(1)}{\Rightarrow}\partial_\mu\partial^\mu\phi-\frac{\partial\mathcal{L}_S}{\partial\phi}=0\overset{(2)}{\Rightarrow}$ $\partial_\mu\partial^\mu\phi-(-m^2\phi)=0\Longrightarrow\partial_\mu\partial^\mu\phi+m^2\phi=0$ ... Klein-Gordon equation for a free scalar field
Relativistic (spin half) spinor fields	$\mathcal{L}$ for free-particle Dirac equation $\mathcal{L}_D=\bar\Psi(i\gamma^\mu\partial_\mu-m)\Psi$ $\quad=i\bar\Psi\gamma^\mu\partial_\mu\Psi-m\bar\Psi\Psi$ ... (3) | Euler-Lagrange $\partial_\mu\left(\frac{\partial\mathcal{L}_D}{\partial(\partial_\mu\bar\Psi)}\right)-\frac{\partial\mathcal{L}}{\partial\bar\Psi}=0$ ... (4) $\frac{\partial\mathcal{L}_D}{\partial(\partial_\mu\bar\Psi)}\overset{(3)}{=}0\overset{(4)}{\Rightarrow}-\frac{\partial\mathcal{L}}{\partial\bar\Psi}=0\overset{(3)}{\Rightarrow}-(i\gamma^\mu\partial_\mu\Psi-m\Psi)=0\Rightarrow$ $(i\gamma^\mu\partial_\mu-m)\Psi=0$ ... Dirac equation
Relativistic EM vector field	$\mathcal{L}$ for electromagnetic field $\mathcal{L}_{EM}=-\frac14F^{\mu\nu}F_{\mu\nu}-j^\mu A_\mu$ ... (5) $F^{\mu\nu}\overset{\text{def}}{=}\begin{pmatrix}0&-E_x&-E_y&-E_z\\E_x&0&-B_z&B_y\\E_y&B_z&0&-B_x\\E_z&-B_y&B_x&0\end{pmatrix}$ $A^\mu\overset{\text{def}}{=}\begin{pmatrix}\phi\\\vec A\end{pmatrix}$ $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ Maxwell: $\partial_\mu F^{\mu\nu}=j^\nu$ $A_\mu\to A'_\mu=A_\mu-\partial_\mu\chi$ | Euler-Lagrange $\partial_\nu\left(\frac{\partial\mathcal{L}_{EM}}{\partial(\partial_\nu A_\mu)}\right)-\frac{\partial\mathcal{L}}{\partial A_\mu}=0$ ... (6)
Deriving Maxwell's equation from $\mathcal{L}_{EM}$	$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\overset{(5)}{\Rightarrow}\mathcal{L}_{EM}=-\frac14(\partial^\mu A^\nu-\partial^\nu A^\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)-j^\mu A_\mu$ $\mathcal{L}_{EM}=-\frac14\underbrace{\partial^\mu A^\nu\partial_\mu A_\nu}_{\mu\leftrightarrow\nu}-\frac14\partial^\nu A^\mu\partial_\nu A_\mu+\frac14\underbrace{\partial^\mu A^\nu\partial_\nu A_\mu}_{\mu\leftrightarrow\nu}+\frac14\partial^\nu A^\mu\partial_\mu A_\nu-j^\mu A_\mu$ $\mathcal{L}_{EM}=-\frac14\partial^\nu A^\mu\partial_\nu A_\mu-\frac14\partial^\nu A^\mu\partial_\nu A_\mu+\frac14\partial^\nu A^\mu\partial_\mu A_\nu+\frac14\partial^\nu A^\mu\partial_\mu A_\nu-j^\mu A_\mu\Rightarrow\mathcal{L}_{EM}=-\frac12\partial^\nu A^\mu\partial_\nu A_\mu+\frac12\partial^\nu A^\mu\partial_\mu A_\nu-j^\mu A_\mu$ ... (7) $\frac{\partial\mathcal{L}_{EM}}{\partial(\partial_\nu A_\mu)}\overset{(7)}{=}-\frac12\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial^\nu_{\downarrow}A^\mu_{\downarrow})\partial_\nu^{\uparrow}A_\mu^{\uparrow}-\frac12\partial^\nu A^\mu\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\nu A_\mu)+\frac12\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial^\nu_{\downarrow}A^\mu_{\downarrow})\partial_\mu^{\uparrow}A_\nu^{\uparrow}+\underbrace{\frac12\partial^\nu A^\mu\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\mu A_\nu)}_{\mu\leftrightarrow\nu}$ $\frac{\partial\mathcal{L}_{EM}}{\partial(\partial_\nu A_\mu)}=-\frac12\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\nu A_\mu)\partial^\nu A^\mu-\frac12\partial^\nu A^\mu\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\nu A_\mu)+\frac12\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\nu A_\mu)\partial^\mu A^\nu+\frac12\partial^\mu A^\nu\frac{\partial}{\partial(\partial_\nu A_\mu)}(\partial_\nu A_\mu)$ $\frac{\partial\mathcal{L}_{EM}}{\partial(\partial_\nu A_\mu)}=-\frac12\partial^\nu A^\mu-\frac12\partial^\nu A^\mu+\frac12\partial^\mu A^\nu+\frac12\partial^\mu A^\nu\Longrightarrow\frac{\partial\mathcal{L}_{EM}}{\partial(\partial_\nu A_\mu)}=-\partial^\nu A^\mu+\partial^\mu A^\nu\overset{(6)}{\Rightarrow}\partial_\nu(-\partial^\nu A^\mu+\partial^\mu A^\nu)-\frac{\partial\mathcal{L}}{\partial A_\mu}=0\overset{(7)}{\Rightarrow}$ $\partial_\nu(-\partial^\nu A^\mu+\partial^\mu A^\nu)-(-j^\mu)=0\Longrightarrow j^\mu=\partial_\nu(\partial^\nu A^\mu-\partial^\mu A^\nu)\Longrightarrow\underbrace{j^\mu=\partial_\nu F^{\nu\mu}}_{\nu\leftrightarrow\mu}\Longrightarrow\partial_\mu F^{\mu\nu}=j^\nu$ ... Maxwell's equation
Massive spin 1 particle	$\mathcal{L}_{Proca}=-\frac14F^{\mu\nu}F_{\mu\nu}+\frac12m_\gamma^2A^\mu A_\mu$

Local Gauge-Invariance leads to QED Lagrangian

Local Phase transformation	Requiring the Dirac equation to be invariant under a U(1) *local* phase transformation introduces the electromagnetic interaction. The required gauge symmetry is expressed naturally as the invariance of the Lagrangian under a local phase transformation of the fields: $\Psi(x^\alpha)\to\Psi'(x^\alpha)=e^{iq\,\chi(x^\alpha)}\,\Psi(x^\alpha)$ ... (1)
$\mathcal{L}_D$ is not invariant to local phase transformations	$\mathcal{L}_D=i\bar\Psi\gamma^\mu\partial_\mu\Psi-m\bar\Psi\Psi$ ... (2) $\to\mathcal{L}'_D=i\bar\Psi'\gamma^\mu\partial_\mu\Psi'-m\bar\Psi'\Psi'\overset{(1)}{\Rightarrow}$ $\mathcal{L}'_D=ie^{-iq\chi}\bar\Psi\gamma^\mu\partial_\mu(e^{iq\chi}\Psi)-m\cancel{e^{-iq\chi}}\bar\Psi\cancel{e^{iq\chi}}\Psi=e^{-iq\chi}\bar\Psi\gamma^\mu\left((\partial_\mu e^{iq\chi})\Psi+e^{iq\chi}\partial_\mu\Psi\right)-m\bar\Psi\Psi$ $\mathcal{L}'_D=ie^{-iq\chi}\bar\Psi\gamma^\mu(iq(\partial_\mu\chi)e^{iq\chi}\Psi+e^{iq\chi}\partial_\mu\Psi)-m\bar\Psi\Psi=-\cancel{e^{-iq\chi}}\bar\Psi\gamma^\mu q(\partial_\mu\chi)\cancel{e^{iq\chi}}\Psi+i\cancel{e^{-iq\chi}}\bar\Psi\gamma^\mu\cancel{e^{iq\chi}}\partial_\mu\Psi-m\bar\Psi\Psi$ $\mathcal{L}'_D=i\bar\Psi\gamma^\mu\partial_\mu\Psi-m\bar\Psi\Psi-q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi\overset{(2)}{\Rightarrow}\mathcal{L}'_D=\mathcal{L}_D-q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi$ ... (2) ... not invariant to local phase transformations
gauge invariant $\mathcal{L}$ for spin-half fermion	The required gauge-invariance can be restored by replacing the derivative ∂_μ with the covariant derivative D_μ: $\partial_\mu\to D_\mu=\partial_\mu+iqA_\mu$ ... (3) where A_μ is a new gauge field that transforms as $A_\mu\to A'_\mu=A_\mu-\partial_\mu\chi$ ... (4) (2)(3) $\Longrightarrow\mathcal{L}_{inv}=i\bar\Psi\gamma^\mu D_\mu\Psi-m\bar\Psi\Psi\overset{(3)}{\Rightarrow}\mathcal{L}_{inv}=i\bar\Psi\gamma^\mu(\partial_\mu+iqA_\mu)\Psi-m\bar\Psi\Psi=i\bar\Psi\gamma^\mu\partial_\mu\Psi-q\bar\Psi\gamma^\mu A_\mu\Psi-m\bar\Psi\Psi\Longrightarrow$ $\mathcal{L}_{inv}=\bar\Psi(i\gamma^\mu\partial_\mu-m)\Psi-q\bar\Psi\gamma^\mu A_\mu\Psi$ ... (5) $\overset{(2)}{\Rightarrow}\mathcal{L}_{inv}=\mathcal{L}_D-q\bar\Psi\gamma^\mu\Psi A_\mu$ ... (6) Interaction of the fermion with the photon
$\mathcal{L}_{inv}$ is invariant to gauge transform.	$\mathcal{L}'_{inv}=\mathcal{L}'_D-q\bar\Psi'\gamma^\mu A'_\mu\Psi'\overset{(2)}{\Rightarrow}\mathcal{L}'_{inv}=\mathcal{L}_D-q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi-q\bar\Psi'\gamma^\mu A'_\mu\Psi'\overset{(1)}{\Rightarrow}$ $\mathcal{L}'_{inv}=\mathcal{L}_D-q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi-q\cancel{e^{-iq\chi}}\bar\Psi\gamma^\mu A'_\mu\cancel{e^{iq\chi}}\Psi\overset{(4)}{\Rightarrow}\mathcal{L}'_{inv}=\mathcal{L}_D-q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi-q\bar\Psi\gamma^\mu(A_\mu-\partial_\mu\chi)\Psi$ $\mathcal{L}'_{inv}=\mathcal{L}_D-\cancel{q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi}-q\bar\Psi\gamma^\mu A_\mu\Psi+\cancel{q\bar\Psi\gamma^\mu(\partial_\mu\chi)\Psi}=\mathcal{L}_D-q\bar\Psi\gamma^\mu\Psi A_\mu\overset{(2)}{\Rightarrow}\mathcal{L}'_{inv}=\mathcal{L}_{inv}$
$\mathcal{L}_{QED}$	Considering that $F^{\mu\nu}F_{\mu\nu}$ is already invariant under the transformation $A_\mu\to A'_\mu=A_\mu-\partial_\mu\chi$, we can write the complete QED Lagrangian, describing the fields for the electron (with $q=-e$), the massless photon and the interaction between them as $\mathcal{L}_{QED}=\underbrace{i\bar\Psi\gamma^\mu\partial_\mu\Psi}_{\text{kinetic term electron}}-\underbrace{m_e\bar\Psi\Psi}_{\text{electron mass term}}+\underbrace{e\bar\Psi\gamma^\mu\Psi A_\mu}_{\text{potential term } e^-\gamma \text{ interaction}}-\underbrace{\frac14F^{\mu\nu}F_{\mu\nu}}_{\text{kinetic term photon}}$ with $j^\mu=-e\bar\Psi\gamma^\mu\Psi\Longrightarrow$ $\mathcal{L}_{QED}=i\bar\Psi\gamma^\mu\partial_\mu\Psi-m_e\bar\Psi\Psi-j^\mu A_\mu-\frac14F^{\mu\nu}F_{\mu\nu}=\mathcal{L}_D+\mathcal{L}_{EM}$

The Higgs Mechanism - Introduction

Problem: particle masses break gauge invariance

If the photon were massive, the Lagrangian of the QED would contain an additional term $\frac{1}{2}m_\gamma^2 A^\mu A_\mu$

$$\mathcal{L}_{QED}^{massive} = \underbrace{i\bar{\Psi}\gamma^\mu\partial_\mu\Psi}_{\text{kinetic term electron}} - m\bar{\Psi}\Psi + \underbrace{e\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\text{potential term } e^-\gamma \text{ interaction}} - \underbrace{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{kinetic term photon}} + \underbrace{\frac{1}{2}m_\gamma^2 A^\mu A_\mu}_{\text{photon mass}}$$

But the new mass-term is not gauge invariant: $\frac{1}{2}m_\gamma^2 A^\mu A_\mu \to \frac{1}{2}m_\gamma^2(A^\mu - \partial^\mu\chi)(A_\mu - \partial_\mu\chi) \neq \frac{1}{2}m_\gamma^2 A^\mu A_\mu$

This problem is not limited to the $U(1)$ local gauge symmetry of QED, it also applies to the $SU(3)$ gauge symmetries of QCD and the $SU(2)_L$ gauge symmetry of the weak interaction. This is a problem for the large masses of the W and Z bosons.

The problem is not restricted to the gauge bosons. The electron mass term $m_e\bar{\Psi}\Psi$ in the QED Lagrangian can be written as:

$$m_e\bar{\Psi}\Psi = m_e\bar{\Psi}\left(\frac{1}{2}(1-\gamma^5)\Psi + \frac{1}{2}(1+\gamma^5)\Psi\right) = m_e\bar{\Psi}\left(\frac{1}{2}\Psi_L + \frac{1}{2}\Psi_R\right)\Bigg| \begin{matrix}(1-\gamma^5)\Psi_L = \Psi_L \\ (1+\gamma^5)\Psi_R = \Psi_R\end{matrix} \Longrightarrow$$

$$m_e\bar{\Psi}\Psi = m_e\bar{\Psi}\left(\frac{1}{2}(1-\gamma^5)\Psi_L + \frac{1}{2}(1+\gamma^5)\Psi_R\right) = \frac{1}{2}m_e(\bar{\Psi}(1-\gamma^5)\Psi_L + \bar{\Psi}(1+\gamma^5)\Psi_R) = \frac{1}{2}m_e(\bar{\Psi}_R\Psi_L + \bar{\Psi}_L\Psi_R)$$

In the $SU(2)_L$ gauge transformation of the weak interaction, left-handed particles transform as weak isospin doublets, and right-handed particles as singlets, and therefore the mass term of the electron also breaks the gauge invariance.

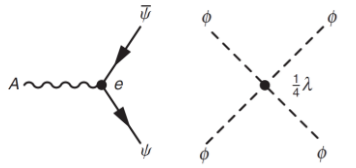
Interacting scalar field

Consider the scalar field ϕ with the potential $\boxed{V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4}$ ... (1)

The corresponding Lagrangian is: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \Longrightarrow$

$$\boxed{\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi}_{\text{kinetic energy of particle}} - \underbrace{\frac{1}{2}\mu^2\phi^2}_{\text{particle mass}^*} - \underbrace{\frac{1}{4}\lambda\phi^4}_{\text{self-interaction}}} \text{ ... (2)}$$

*...for $\mu^2 > 0$

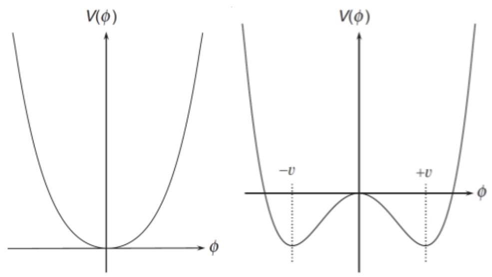


symmetry breaking

The vacuum state is the lowest energy state of ϕ and corresponds to the minumum of $V(\phi)$. For $V(\phi)$ to have a minimum: $\lambda > 0$.

If $\mu^2 > 0 \Longrightarrow$ $V(\phi)$ has a minimum at $\phi = 0 \Longrightarrow$
Vacuum state at $\phi = 0$, scalar particle with mass μ,
four-point self-interaction proportional to ϕ^4

If $\mu^2 < 0 \Longrightarrow$ $V(\phi)$ has a minimum at $\phi_{min} = \pm v = \pm\sqrt{\frac{-\mu^2}{\lambda}}$* $\Longrightarrow$
μ can no longer be interpreted as mass, there are two degenerate vacuum states at $\phi = \pm v$. Choice of vacuum state breaks symmetry of $\mathcal{L} \Longrightarrow$ “spontaneous symmetry breaking”



*calulating $\phi_{min} = \pm v$ for $\mu^2 < 0$: $V'(v) = 0 \overset{(1)}{\Longrightarrow} \mu^2 v + \lambda v^3 = 0 \Longrightarrow \mu^2 + \lambda v^2 = 0 \Longrightarrow \lambda v^2 = -\mu^2 \Longrightarrow v = \pm\sqrt{\frac{-\mu^2}{\lambda}}$ ... (3)

getting $\mathcal{L}(\eta)$ from perturbations around the vacuum state

If the vacuum state of the scalar field is chosen to be at $\phi = +v$, the excitations of the field, which describes the particle state, can be obtained by considering perturbations of the field ϕ around the vacuum state: $\phi(x) = v + \eta(x)$ ... (4)

Because the vacuum state $v = \text{const.} \overset{(4)}{\Longrightarrow} \partial_\mu\phi = \partial_\mu\eta$ ... (5) By inserting (4) and (5) into (2) we get:

$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}\mu^2(v+\eta)^2 - \frac{1}{4}\lambda(v+\eta)^4 \Big| v = \sqrt{\frac{-\mu^2}{\lambda}} \Longrightarrow v^2 = \frac{-\mu^2}{\lambda} \Longrightarrow -\mu^2 = \lambda v^2 \Longrightarrow \mu^2 = -\lambda v^2 \text{ ... (6)}$$

$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\lambda v^2(v^2 + \eta^2 + 2v\eta) - \frac{1}{4}\lambda(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4)$$

$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\lambda v^4 + \frac{1}{2}\lambda v^2\eta^2 + \cancel{\lambda v^3\eta} - \frac{1}{4}\lambda v^4 - \cancel{\lambda v^3\eta} - \frac{3}{2}\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4$$

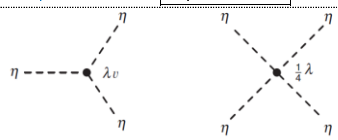
$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{4}\lambda v^4 - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 \Big| \frac{1}{4}\lambda v^4 \text{ is constant and therefore irrelevant} \Longrightarrow$$

$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 \text{ ... (7)}$$

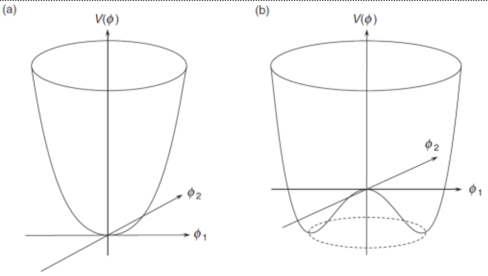
The term $\lambda v^2\eta^2$ corresponds to the original mass term $\frac{1}{2}\mu^2\phi^2 \Longrightarrow$

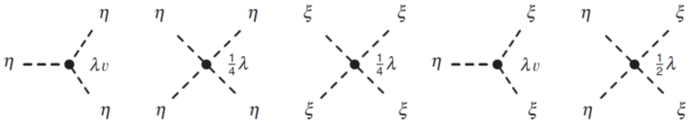
$$\mathcal{L}(\eta) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}m_\eta^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 \Big| \text{with } \frac{1}{2}m_\eta^2 = \lambda v^2 \Longrightarrow m_\eta^2 = 2\lambda v^2 \overset{(6)}{\Longrightarrow} m_\eta^2 = -2\mu^2 \Longrightarrow \boxed{m_\eta = \sqrt{-2\mu^2}}$$

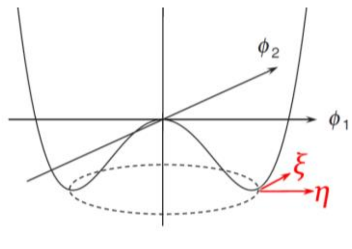
$$\boxed{\mathcal{L}(\eta) = \underbrace{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta}_{\text{kinetic enery of particle}} - \underbrace{\frac{1}{2}m_\eta^2\eta^2}_{\text{mass term}} - V(\eta) \text{ with } V(\eta) = \underbrace{\lambda v\eta^3}_{\text{triple interact.}} + \underbrace{\frac{1}{4}\lambda\eta^4}_{\text{quartic interact.}}}$$



Symmetry breaking for a complex scalar field

<table>
<tr><td>Lagrangian</td><td>
The idea of spontaneous symmetry breaking is now applied to a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ ... (1)
The corresponding Lagrangian is: $\boxed{\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - V(\phi) \text{ with } V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2 \text{ and } \lambda > 0}$ ... (2)
This special form becomes clear when $\mathcal{L}$ is expressed in terms of ϕ_1 and ϕ_2:
$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - (\phi^*\phi) - \lambda(\phi^*\phi)^2 \,|\, \phi^*\phi = |\phi|^2 \overset{(1)}{=} \frac{1}{2}(\phi_1^2 + \phi_2^2)$
$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2 \overset{(1)}{\Longrightarrow}$
$\mathcal{L} = \frac{1}{\sqrt{2}}(\partial_\mu\phi_1 - i\partial_\mu\phi_2)\frac{1}{\sqrt{2}}(\partial^\mu\phi_1 + i\partial^\mu\phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$
$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + i\partial_\mu\phi_1\partial^\mu\phi_2 - i\partial^\mu\phi_1\partial_\mu\phi_2 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$
$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \cancel{i\partial_\mu\phi_1\partial^\mu\phi_2 - i\partial_\mu\phi_1\partial^\mu\phi_2} + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$
$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2}$ ... (3)
</td></tr>
<tr><td>symmetry breaking</td><td>
The vacuum state is the lowest energy state of ϕ and corresponds to the minumum of $V(\phi)$. For $V(\phi)$ to have a minimum: $\lambda > 0$.
If $\mu^2 > 0 \Longrightarrow$ $V(\phi)$ has a minimum at $\phi_1 = \phi_2 = 0 \Longrightarrow$
Vacuum state at $\phi_1 = \phi_2 = 0$
If $\mu^2 < 0 \Longrightarrow$ $V(\phi)$ has a minimum at $\phi_1^2 + \phi_2^2 = v^2 = \frac{-\mu^2}{\lambda}$* $\Longrightarrow$
indicated by the circle in image. The physical vacuum state corresponds to a paticular point on the circle, breaking the global U(1) symmetry of $\mathcal{L}$. $\Longrightarrow$ "spontaneous symmetry breaking"


*calculating v^2: $\phi^*\phi \overset{\text{def}}{=} s \overset{(2)}{\Rightarrow} V = \mu^2 s + \lambda s^2$ ... (4) $V'(s) = 0 \overset{(4)}{\Rightarrow} \mu^2 + 2\lambda s = 0 \Rightarrow 2\lambda s = -\mu^2 \Rightarrow s = \frac{-\mu^2}{2\lambda} \,|\, s = \phi^*\phi \Rightarrow$
$\phi^*\phi = \frac{-\mu^2}{2\lambda} \,|\, \phi^*\phi = |\phi|^2 \overset{(1)}{=} \frac{1}{2}(\phi_1^2 + \phi_2^2) \Rightarrow \frac{1}{2}(\phi_1^2 + \phi_2^2) = \frac{-\mu^2}{2\lambda} \Rightarrow \boxed{\phi_1^2 + \phi_2^2 = \frac{-\mu^2}{\lambda} = v^2}$ ... (5a) $\Rightarrow \mu^2 = -\lambda v^2$ ... (5b)
</td></tr>
<tr><td>getting $\mathcal{L}(\eta, \xi)$ from perturbations around the vacuum state and creating the Goldstone boson</td><td>
Without l.o.g. the vacuum state can be chosen to be in real direction $(\phi_1, \phi_2) = (v, 0)$, and the complex scalar field ϕ can be expanded by considering perturbations $\eta(x)$ and $\xi(x)$ around the vacuum state: $\boxed{\phi_1(x) = v + \eta(x), \phi_2(x) = \xi(x)}$ ... (6)
Because the vacuum state $v = $ const. $\overset{(6)}{\Rightarrow}$ $\partial_\mu\phi = \partial_\mu\eta = \partial_\mu\xi$ ... (7) By inserting (6) and (7) into (4) we get:
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \frac{1}{2}\mu^2((v+\eta)^2 + \xi^2) - \frac{1}{4}\lambda((v+\eta)^2 + \xi^2)^2 \overset{(5b)}{\Longrightarrow}$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}\lambda v^2((v+\eta)^2 + \xi^2) - \frac{1}{4}\lambda((v+\eta)^2 + \xi^2)^2$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}\lambda v^2(v^2 + \eta^2 + 2v\eta + \xi^2) - \frac{1}{4}\lambda(v^2 + \eta^2 + 2v\eta + \xi^2)^2$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi +$
$\frac{1}{2}\lambda v^4 + \frac{1}{2}\lambda v^2\eta^2 + \lambda v^3\eta + \frac{1}{2}\lambda v^2\xi^2 - \frac{1}{4}\lambda(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4 + 2v^2\xi^2 + 4v\eta\xi^2 + 2\eta^2\xi^2 + \xi^4)$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi +$
$\frac{1}{2}\lambda v^4 + \frac{1}{2}\lambda v^2\eta^2 + \cancel{\lambda v^3\eta} + \cancel{\frac{1}{2}\lambda v^2\xi^2} - \frac{1}{4}\lambda v^4 - \cancel{\lambda v^3\eta} - \frac{3}{2}\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 - \cancel{\frac{1}{2}\lambda v^2\xi^2} - \lambda v\eta\xi^2 - \frac{1}{2}\lambda\eta^2\xi^2 - \frac{1}{4}\lambda\xi^4$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{4}\lambda v^4 - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\xi^4 - \frac{1}{4}\lambda\eta^4 - \lambda v\eta\xi^2 - \frac{1}{2}\lambda\eta^2\xi^2 \,|\, \lambda v^4 = const \Longrightarrow$ irrelevant
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\xi^4 - \frac{1}{4}\lambda\eta^4 - \lambda v\eta\xi^2 - \frac{1}{2}\lambda\eta^2\xi^2 \overset{(5b)}{\Longrightarrow}$
$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \mu^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\xi^4 - \frac{1}{4}\lambda\eta^4 - \lambda v\eta\xi^2 - \frac{1}{2}\lambda\eta^2\xi^2 \,|$ let's express the mass term $+\mu^2\eta^2$ as $-\frac{1}{2}m_\eta^2\eta^2$
$\boxed{\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi}_{\text{kinetic energy}} - \underbrace{\frac{1}{2}m_\eta^2\eta^2}_{\text{mass term}} - \underbrace{\lambda v\eta^3}_{(a)} - \underbrace{\frac{1}{4}\lambda\eta^4}_{(b)} - \underbrace{\frac{1}{4}\lambda\xi^4}_{(c)} - \underbrace{\lambda v\eta\xi^2}_{(d)} - \underbrace{\frac{1}{2}\lambda\eta^2\xi^2}_{(e)}}$ $\,|\, -\frac{1}{2}m_\eta^2\eta^2 = \mu^2\eta^2 \Longrightarrow \boxed{m_\eta = \sqrt{-2\mu^2}}$


(a) (b) (c) (d) (e)
The excitiations of the massive field η are in the ϕ_1-direction where the potential $V(\phi)$ is (to the first order) quadratic. In contrast, the particles described by the massless field ξ correspond to excitations in ϕ_2-direction, where the potential does not change. This massless sclar particle is known as the Goldstone boson.


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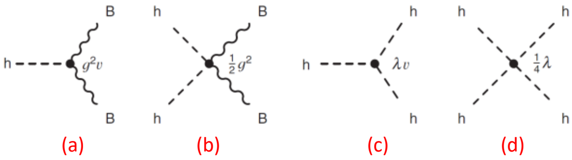
</table>

The Higgs Mechanism

General	In the Higgs mechanism, the spontaneous symmetry breaking of a complex scalar field $V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$ is embedded in a theory with *local* gauge symmetry. In this example, we will use U(1) local gauge symmetry.
Invariance under local gauge trafo	The Lagrangian for a complex scalar field $\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - V(\phi)$ with $V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2$ is *not* invariant under the U(1) local gauge transformation $\phi(x) \to \phi'(x) = \phi(x)\, e^{ig\,\chi(x)}$. This is because of the derivatives $(\partial_\mu\phi)^*(\partial^\mu\phi)$ in the Lagrangian. The required U(1) local gauge invariance can be achieved by replacing the derivative ∂_μ with the derivative $\boxed{\partial_\mu \to D_\mu = \partial_\mu + igB_\mu}$ ... (1). The resulting Lagrangian $\mathcal{L}_{inv} = (D_\mu\phi)^*(D^\mu\phi) - V(\phi)$ is gauge invariant, provided the new gauge field B_μ transforms as $\boxed{B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)}$. This is then the combined Lagrangian for the complex scalar field ϕ and the gauge field B_μ: $\boxed{\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \text{ with } F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu}$ $\overset{(1)}{\Rightarrow}$ $\mathcal{L} = \left((\partial_\mu + igB_\mu)\phi\right)^*\left((\partial^\mu + igB^\mu)\phi\right) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ $\mathcal{L} = (\partial_\mu\phi + igB_\mu\phi)^*(\partial^\mu\phi + igB^\mu\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ $\mathcal{L} = \left((\partial_\mu\phi)^* - igB_\mu\phi^*\right)(\partial^\mu\phi + igB^\mu\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ $\boxed{\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \underbrace{-igB_\mu\phi^*(\partial^\mu\phi) + ig(\partial_\mu\phi)^*B^\mu\phi + g^2B_\mu B^\mu\phi^*\phi}_{\text{additional Terms from } \partial_\mu \to D_\mu = \partial_\mu + igB_\mu}}$ ... (2)
symmetry breaking	Again, for $\mu^2 < 0$ the vacuum state is degenerate and the choice of the physical vacuum state spontaneously breaks the the symmtry of the Lagrangian. As before, the vacuum state can be chosen w.l.o.g. to be in real direction $(\phi_1, \phi_2) = (v, 0)$, and the complex scalar field ϕ can be expanded by considering perturbations $\eta(x)$ and $\xi(x)$ around the vacuum state: $\boxed{\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\,\xi(x))}$ ... (3) Substituting this into (2) leads to $\boxed{\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2}_{\text{massive } \eta} + \underbrace{\frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)}_{\text{massless } \xi \text{ (Goldstone)}} \underbrace{- \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}g^2v^2B_\mu B^\mu}_{\text{massive gauge field } B_\mu} \underbrace{-V_{int}}_{\eta B \text{ interactions}} + \underbrace{gvB_\mu\partial^\mu\xi}_{\xi B \text{ interaction}}}$ ... (4) The previously massless gauge field B_μ has acquired a mass term $\frac{1}{2}g^2v^2B_\mu B^\mu$, achieving the aim of giving a mass to the gauge boson of the local gauge symmetry.
Degrees of freedom problem	The original Lagrangian contained four degrees of freedom: One for each of the scalar fields ϕ_1 and ϕ_2, and two transverse polarisation states for the massless gauge field B_μ. In the Lagrangian (4) the gauge boson has become massive and therefore has the additional longitudinal polarization state. An additional degree of freedom seems to have appeared. Furthermore, the $gvB_\mu\partial^\mu\xi$ term appears to represent a direct coupling between the Goldstone field ξ and the gauge field B_μ.
Eliminating the Gold-stone boson by gauge choice	To solve this, we re-write the Lagrangian (4), starting with just re-arranging the terms: $\mathcal{L} = \frac{1}{2}g^2v^2B_\mu B^\mu + \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + gvB_\mu\partial^\mu\xi + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V_{int}$ $\mathcal{L} = \frac{1}{2}g^2v^2(B_\mu)^2 + \frac{1}{2}(\partial_\mu\xi)^2 + gvB_\mu(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V_{int}$ $\mathcal{L} = \frac{1}{2}g^2v^2\left((B_\mu)^2 + \frac{1}{g^2v^2}(\partial_\mu\xi)^2 + 2\frac{1}{gv}B_\mu(\partial^\mu\xi)\right) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V_{int}$ $\mathcal{L} = \frac{1}{2}g^2v^2\Big(\underbrace{B_\mu + \frac{1}{gv}(\partial_\mu\xi)}_{\stackrel{\text{def}}{=} B'_\mu}\Big)^2 + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V_{int}$ $\vert$ gauge transformation: $B_\mu \to B'_\mu = B_\mu + \frac{1}{gv}(\partial_\mu\xi)$ $\boxed{\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2}_{\text{massive scalar Higgs field } \eta} \underbrace{- \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}g^2v^2B'_\mu B'^\mu}_{\text{massive gauge boson } B_\mu} \underbrace{-V_{int}}_{\eta B \text{ interactions}}}$ ... (5) By choosing the gauge transformation $B_\mu \to B'_\mu = B_\mu + \frac{1}{gv}(\partial_\mu\xi)$ we eliminated the the Goldstone boson. (“The Goldstone ξ has been eaten by the gauge field B'_μ”). This transformation is equivalent to $B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)$ with $\boxed{\chi(x) \stackrel{\text{def}}{=} -\frac{\xi(x)}{gv}}$ ... (6) The corresponding gauge transformation of $\phi(x)$ is therefore $\boxed{\phi(x) \to \phi'(x) = \phi(x)\, e^{-ig\frac{\xi(x)}{gv}} = \phi(x)\, e^{-i\,\xi(x)/v}}$ ... (7)
Unitary gauge	$f(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\,\xi(x)/v}$ ... (8) $\vert$ η and ξ are just small excitations $\Longrightarrow e^{i\,\xi(x)/v} \approx 1 + \frac{i\,\xi(x)}{v} \Longrightarrow$ $f(x) = \frac{1}{\sqrt{2}}(v + \eta(x))\left(1 + \frac{i\,\xi(x)}{v}\right) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\,\xi(x) + \frac{i\,\eta(x)\,\xi(x)}{v}\right)$ $\vert$ $\eta(x)\,\xi(x) \approx 0 \Longrightarrow$ $f(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\,\xi(x)) \overset{(3)}{\Rightarrow} f(x) = \phi(x) \overset{(8)}{\Rightarrow} \phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\,\xi(x)/v} \overset{(7)}{\Rightarrow} \phi'(x) = \frac{1}{\sqrt{2}}(v + \eta(x))\cancel{e^{i\,\xi(x)/v}e^{-i\,\xi(x)/v}} \Longrightarrow$ $\phi'(x) = \frac{1}{\sqrt{2}}(v + \eta(x))$ $\vert$ $\eta(x) \stackrel{\text{def}}{=} h(x)$ Higgs $\boxed{\phi'(x) = \frac{1}{\sqrt{2}}(v + h(x))}$ ... (9) $\overset{(2)}{\Rightarrow}$

Lagrangian of a massive scalar Higgs field h and a massive gauge boson B_μ	$\mathcal{L} = \left(\partial_\mu\phi'\right)^*(\partial^\mu\phi') - \mu^2(\phi'^*\phi') - \lambda(\phi'^*\phi')^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - igB_\mu\phi'^*(\partial^\mu\phi') + ig\left(\partial_\mu\phi'\right)^*B^\mu\phi' + g^2B_\mu B^\mu\phi'^*\phi' \Big\vert\ \phi' \in \mathbb{R}$ $\mathcal{L} = \partial_\mu\phi'\partial^\mu\phi' - \mu^2\phi'^2 - \lambda\phi'^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - igB_\mu\phi'\partial^\mu\phi' + igB^\mu_\downarrow\phi'\partial^\uparrow_\mu\phi' + g^2B_\mu B^\mu\phi'^2$ $\mathcal{L} = \partial_\mu\phi'\partial^\mu\phi' - \mu^2\phi'^2 - \lambda\phi'^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \cancel{igB_\mu\phi'\partial^\mu\phi' + igB_\mu\phi'\partial^\mu\phi'} + g^2B_\mu B^\mu\phi'^2 \overset{(9)}{\Rightarrow}$ $\mathcal{L} = \frac{1}{2}\partial_\mu(v+h)\partial^\mu(v+h) - \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 \Big\vert\ \partial_\mu(v+h) = \partial_\mu h$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 \Big\vert$ re-arrange $\Longrightarrow$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4 \Big\vert\ \mu^2 = -\lambda v^2$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\lambda v^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\lambda v^2(v^2+h^2+2vh) - \frac{1}{4}\lambda(v^4+4v^3h+6v^2h^2+4vh^3+h^4)$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\lambda v^2(v^2+h^2+2vh) - \frac{1}{4}\lambda(v^4+4v^3h+6v^2h^2+4vh^3+h^4)$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \underline{\frac{1}{2}\lambda v^4} + \underline{\frac{1}{2}\lambda v^2h^2} + \cancel{\lambda v^3h} - \underline{\frac{1}{4}\lambda v^4} - \cancel{\lambda v^3h} - \underline{\frac{3}{2}\lambda v^2h^2} - \lambda vh^3 - \frac{1}{4}\lambda h^4$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v+h)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}\lambda v^4 - \lambda v^2h^2 - \lambda vh^3 - \frac{1}{4}\lambda h^4 \Big\vert\ \frac{1}{4}\lambda v^4 = const \Longrightarrow$ irrelevant $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2B_\mu B^\mu(v^2+h^2+2vh) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \lambda v^2h^2 - \lambda vh^3 - \frac{1}{4}\lambda h^4$ $\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2v^2B_\mu B^\mu + \frac{1}{2}g^2h^2B_\mu B^\mu + g^2vB_\mu B^\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \lambda v^2h^2 - \lambda vh^3 - \frac{1}{4}\lambda h^4 \Big\vert$ re-arrange$\Longrightarrow$ $\boxed{\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu h\partial^\mu h - \overbrace{\lambda v^2}^{\frac{1}{2}m_H^2} h^2}_{\text{massive scalar Higgs}} \underbrace{- \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overbrace{\frac{1}{2}g^2v^2}^{\frac{1}{2}m_B^2} B_\mu B^\mu}_{\text{massive gauge boson}} + \underbrace{\overbrace{g^2vB_\mu B^\mu}^{(a)} + \overbrace{\frac{1}{2}g^2h^2B_\mu B^\mu}^{(b)}}_{h,B_\mu \text{ interactions}} - \underbrace{\overbrace{\lambda vh^3}^{(c)} - \overbrace{\frac{1}{4}\lambda h^4}^{(d)}}_{\text{h self-interaction}}}$ ... (10) This Lagrangian describes a massive scalar Higgs field h and a massive gauge boson B_μ associated with the $\mathrm{U}(1)$ local gauge symmetry. It contains interaction terms between the Higgs boson and the gauge boson, and Higgs self-interaction terms . 
Mass of gauge boson and Higgs Boson	The mass of the gauge boson originates from the mass term $\frac{1}{2}g^2v^2B_\mu B^\mu$ where we identify $\frac{1}{2}g^2v^2$ with $\frac{1}{2}m_B^2$, and hence the mass of the gauge boson $\boxed{m_B = gv}$ is related to the strength of the gauge coupling and the vacuum expectation value v of the Higgs field. The mass of the Higgs boson originates from the mass term λv^2h^2 where we identify λv^2 with $\frac{1}{2}m_H^2$, and hence the mass of the Higgs boson $\boxed{m_H = \sqrt{2\lambda}v}$ is also related to the vacuum expectation value v of the Higgs field.