Geometry and Topology

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Topological Spaces and Homotopy *(Topologische Räume und Homotopie)*

Finite Intersection *("Endliche Schnittmenge")*	The **finite intersection A ∩ B** of two (finite) sets A and B is the set that contains all elements of A that also belong to B (or vice versa), but no other element: $A \cap B = \{x: x \in A \wedge x \in B\}$
Arbitrary Intersection	The **intersection of an arbitrary set of sets** (collection of sets, family of sets) $\{A_i: i \in I\}$ is defined as: $x \in \bigcap_{i \in I} A_i \Leftrightarrow \{x: \forall i \in I: x \in A_i\}$. <u>Attention:</u> If the index set I only contains the empty set ($I = \{\emptyset\}$) then with this definition every possible x satisfies the condition and the intersection is the universal set.
Finite Union *("Endliche Vereinigungsmenge")*	The **finite union A ∩ B** of two (finite) sets A and B is the set of elements which are in A, in B, or both A and B: $A \cup B = \{x: x \in A \vee x \in B\}$
Arbitrary Union *("Endliche Vereinigungsmenge")*	The **union of an arbitrary set of sets** (collection of sets, family of sets) $\{A_i: i \in I\}$ is defined as: $x \in \bigcup_{i \in I} A_i \Leftrightarrow \{x: \exists i \in I: x \in A_i\}$
Carthesian Product *(Karth. Produkt)*	For sets A and B, the **cartesian product** $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$: $A \times B = \{(a, b): a \in A \wedge b \in B\}$
Cartesian square	The **cartesian square** of a set X is the Cartesian product $X^2 = X \times X$. An <u>example</u> is the 2-dimensional plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. $\mathbb{R}^2$ is the set of all points (x, y) where x and y are real numbers (Cartesian coordinate system).
Metric Space *(Metrischer Raum)*	**Metric Space** is a set for which distances between all members of the set are defined. Hence, the Metric Space (X, d) is a set X with a (distance-)function $\mathrm{d}: X \times X \to \mathbb{R}_{\geq 0}$ such that (1) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ (symmetry), (2) $\mathrm{d}(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles), and (3) $\mathrm{d}(x, y) + \mathrm{d}(y, z) \geq \mathrm{d}(x, z)$ (triangle inequality). Remark: Non-negativity $\mathrm{d}(x, y) \geq 0$ follows from (1), (2), (3). <u>Examples:</u> (1) Euclidean Metric on $\mathbb{R}^n$ and subsets, (2) Discrete Metric $\mathrm{d}(x, y) = 1 \; \forall x \neq y$
ε-Neighbourhood *("ε-Umgebung")*	**ε-Neighbourhood** (a.k.a Open Ball Sphere) in metric space (X, d): $U_\varepsilon(x) = \{y \epsilon X: \mathrm{d}(x, y) < \varepsilon\}$ $U_\varepsilon(x)$ is open
Continuity at a point *(Stetigkeit in einem Pkt)*	A map $\mathrm{f}: X \to \tilde{X}$ of metric spaces (X, d) and $(\tilde{X}, \tilde{d})$ is called **continuous at $x \in X$** if for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\tilde{\mathrm{d}}(\mathrm{f}(x), \mathrm{f}(y)) < \varepsilon$ for all y with $\mathrm{d}(x, y) < \delta$ $\forall \varepsilon > 0: \exists \delta > 0: [\mathrm{d}(x, y) < \delta: \tilde{\mathrm{d}}(\mathrm{f}(x), \mathrm{f}(y)) < \varepsilon]$, equivalent to: $\forall \varepsilon > 0: \exists \delta > 0: [y \in U_\delta(x) \Longrightarrow \mathrm{f}(y) \in U_\varepsilon(\mathrm{f}(x))]$
Continuous Map *(Stetige Abbildung)*	A **map** $\mathrm{f}: X \to \tilde{X}$ of metric spaces (X, d) and $(\tilde{X}, \tilde{d})$ is **continuous** if it is continuous at every $x \in X$.
Open Subset *(Offene Teilmenge)*	A subset $\mathcal{O} \subset X$ of a metric space (X, d) is called **open subset** if each of its points has an ε-neighbourhood that is contained in $\mathcal{O}$, i.e. for each of $x \subset \mathcal{O}$ there exists a positive number ε with $U_\varepsilon(x) \subseteq \mathcal{O}$: $\forall x \in \mathcal{O} \; \exists \varepsilon > 0: U_\varepsilon(x) \subseteq \mathcal{O}$. <u>Example</u>: The set of points (x, y) in $\mathbb{R}^2$ $\{(x, y): x^2 + y^2 < r^2\}$
Inverse Image (Preimage) *(Urbild)*	The **inverse image** (or Preimage) of a set $S \subseteq \tilde{X}$ under a function $\mathrm{f}: X \to \tilde{X}$ between metric spaces (X, d) and $(\tilde{X}, \tilde{d})$ is $\mathrm{f}^{-1}[S] \stackrel{\text{def}}{=} \{x \in X: \mathrm{f}(x) \in S\}$
Continous Function *(Stetige Funktion)*	Lemma: A **function** $\mathrm{f}: X \to \tilde{X}$ between metric spaces (X, d) and $(\tilde{X}, \tilde{d})$ is **continuous** if and only if the inverse image $\mathrm{f}^{-1}(\tilde{\mathcal{O}})$ of every open subset $\tilde{\mathcal{O}} \subseteq \tilde{X}$ is an open subset of X.
Powerset *(Potenzmenge)*	The **powerset** $\mathcal{P}(X)$ of any set X is the set of all subsets of X, including the empty set $\emptyset$ and X itself. $\mathcal{P}(X) = \{S: S \subseteq X\}$. <u>Example</u>: If $X = \{x, y, z\}$ then $\mathcal{P}(X) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$
Topology *(Topologie)*	A family $\mathcal{T}$ of subsets of a set X is called **topology** on X if it contains X and the empty set $\emptyset$, as well as finite intersections and arbitrary unions of elements of $\mathcal{T}$. In other words: Let X be a set, and $\mathcal{P}(X)$ a powerset. Then $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a Topology if (1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$ ($\mathcal{T}$ contains X and the empty set), (2) $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots, \mathcal{O}_n \in \mathcal{T} \Longrightarrow \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3 \cap \ldots \cap \mathcal{O}_n \in \mathcal{T}$ ($\mathcal{T}$ contains every finite union of sets $\{\mathcal{O}_i: i \in I\}$), (3) $\mathcal{O}_i \in \mathcal{T} \; \forall i \in I \Longrightarrow \bigcup_{i \in I} \mathcal{O}_i \in \mathcal{T}$ ($\mathcal{T}$ contains every arbitrary union of sets $\{\mathcal{O}_i: i \in I\}$)
Topological Space *(Topologischer Raum)*	If $\mathcal{T}$ is a topology on X, then the pair $(X, \mathcal{T})$ is called a **topological space**. The notation $X_\mathcal{T}$ may be used to denote a set X endowed with the particular topology $\mathcal{T}$.
Continuous Function in $X_\mathcal{T}$ *(Stetige Funktion in $X_\mathcal{T}$)*	Let $(X, \mathcal{T})$ and $(\tilde{X}, \tilde{\mathcal{T}})$ be topological spaces. A function $\mathrm{f}: X \to \tilde{X}$ is called **continuous** if $\mathrm{f}^{-1}(\tilde{\mathcal{O}}) \in \mathcal{T}$ for every $\tilde{\mathcal{O}} \in \tilde{\mathcal{T}}$.
Homeomorphism *(Homöomorphismus)*	A **homeomorphism** is a bijective map f such that both f and f^{-1} are continuous. In such case $(X, \mathcal{T})$ and $(\tilde{X}, \tilde{\mathcal{T}})$ are called homeomorphic.
Induced Topology *(Teilraumtopologie)*	Informally, induced topology (or, Subspace Topology) is the natural structure a subspace of a topological space "inherits" from the topological space. More formally, given a topological space $(X, \mathcal{T}_X)$ and a subset $S \subseteq X$, the **induced topology** (Subspace Topology) $\mathcal{T}_S$ on S is defined by $\mathcal{T}_S \stackrel{\text{def}}{=} \{\mathcal{O} \cap S: \mathcal{O} \in \mathcal{T}_X\}$
Basis of a Topology *(Topologische Basis)*	A Basis (or Base) $\mathcal{B}$ for a topological space X with topology $\mathcal{T}$ is a collection of open sets in $\mathcal{T}$ such that every open set $\mathcal{O}_i$ in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. We say that the base generates the topology $\mathcal{T}$. Hence, a **basis of topology $\mathcal{T}$** is a subset $\mathcal{B}$ of $\mathcal{T}$ such that any $\mathcal{O} \in \mathcal{T}$ can be written as $\mathcal{O} = \{\bigcup_{i \in I} \mathcal{O}_i : \mathcal{O}_i \in \mathcal{B}\}$ <u>Remark:</u> Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology. <u>Examples:</u> (1) Discrete topology: 1-element sets are a basis. (2) Metric Topology: ε-neighbourhoods are a basis: $\mathcal{B} = \{U_\varepsilon(x) : x \in X\}$
Product Topology *(Produkttopologie)*	Given the topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we define $(X \times Y, \mathcal{T}_{X \times Y})$ by taking $\mathcal{B} = \{\mathcal{O}_X \times \mathcal{O}_Y: \mathcal{O}_X \in \mathcal{T}_X, \mathcal{O}_Y \in \mathcal{T}_Y\}$ as a basis for the **product topology** $\mathcal{T}_{X \times Y}$.

Interior *(Inneres, Innerer Kern)*	The **interior** M^0 of a subset M of a topological space X consists of all points of M that do not belong to the boundary of M. Thus, M^0 is the union of all open sets contained in M: $M^0 = \{\bigcup_{i\in I} \mathcal{O}_i : \mathcal{O}_i \in \mathcal{T}, \mathcal{O}_i \subseteq M\}$ The Interior M^0 is defined to be the largest open set contained in M. <u>Example:</u> If M is a ball in $\mathbb{R}^3$ then the Interior M^0 is all points satisfying the inequation $x^2 + y^2 + z^2 < r^2$.
Closure *(Abschluss)*	The closure $\overline{M}$ of a subset M of a topological space X consists of all points in M together with all limit points of M. The **closure** of M may equivalently be defined as the union of M and its boundary, and also as the intersection of all closed sets containing M: $M = \{\bigcap_{i\in I} \mathcal{C}_i : \mathcal{C}_i \supseteq M\}$ Intuitively, the closure can be thought of as all the points that are either in M or "near" M. Example: For $x^2 + y^2 < r^2$ the closure is $x^2 + y^2 \leq r^2$
Dense Subset *(Dichte Teilmenge)*	A subset M of a topological space X is called **Dense** if every point x in X either belongs to M or is a limit point of M. Informally, for every point in X, the point is either in M or arbitrarily "close" to a member of M. $M \subset X$ is called dense in X, if and only if $X = \overline{M}$. <u>Example:</u> Every real number is either a rational number or has one arbitrarily close to it, hence $\mathbb{Q}$ is dense in $\mathbb{R}$.
Boundary *("Rand")*	A **boundary** ∂M of a subset M of a topological space X is the set of points in the closure of M, not belonging to the interior of M: $\partial M = \overline{M} \backslash M^0$. Example: For $x^2 + y^2 < r^2$ the boundary is $x^2 + y^2 = r^2$
Neighbourhood *("Umgebung")*	Let $(X, \mathcal{T})$ be a topological space. For a point $x \in X$ an open subset $\mathcal{O} \in \mathcal{T}$ is called **open neighbourhood** of x if also $x \in \mathcal{O}$. A subset $U \in X$ is called **neighbourhood** of $x \in X$ if $\exists\, \mathcal{O} \in \mathcal{T}: x \in \mathcal{O} \subseteq U$, thus if U contains an open neighbourhood of x. <u>Remark</u>: $S \subseteq X$ is open if and only if S is a neighbourhood of each of its points.
Haussdorff Space *("Haussdorff-Raum")*	Intuitively, a Haussdorff Space is a topological space where all pairs of different points x and y can be separated by neighbourhoods. Formally: A **Haussdorff Space** is a topological space X such that for any $x \in X, y \in X, x \neq y$ there are open sets $\mathcal{O}_1 \ni x, \mathcal{O}_2 \ni y$ so that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. <u>Remark:</u> Almost all spaces encountered in analysis are Hausdorff; most importantly, $\mathbb{R}$ is a Hausdorff space. More generally, all metric spaces are Hausdorff.
Covering ("Abdeckung")	If X is a topological space, then the **covering** C of X is a collection of subsets $S_i \subseteq X$ whose union is the whole space X, thus $X = \bigcup S_i$.
Compact ("kompakt")	A topological space X is called **compact** if, for every covering of X by open sets, a finite number of these sets already constitute a covering. <u>Examples:</u> (1) A closed bounded interval is compact. (2) $\mathbb{R}$ is compact. (3) An open interval is <u>not</u> compact.
Locally Compact ("lokalkompakt")	If X is a topological space, then X is called **locally compact** if every $x \in X$ has a compact neighbourhood.
Theorems about compactness	• A compact subset $S \subseteq X$ of a Hausdorff Space X is closed ("compact⟹closed") • Closed subspaces and continuous images of compact spaces are compact • Metric spaces are compact if and only if every sequence contains a convergent subsequence • For subsets of $\mathbb{R}^n$: (compact)⟺(bounded and closed) • Finite unions of compact spaces are compact
Compactification *("Kompaktifizierung")*	**Compactification** of $(X, \mathcal{T})$ is a compact topological space $(\tilde{X}, \tilde{\mathcal{T}})$ such that $\tilde{X} \supseteq X$, and X is dense in $\tilde{X}$ ($\bar{X} = \tilde{X}$), and $\mathcal{T}$ is the topology that is induced (with respect to the inclusion) on X by $\tilde{\mathcal{T}}$. <u>Example:</u> (1) Compactification of the open ball is the closed ball. (2) Consider the real line $\mathbb{R}$ with its ordinary topology. $\mathbb{R}$ is not compact; in a sense, points can go off to infinity to the left or to the right. It is possible to compactify the real line $\mathbb{R}$ by adding two points, $+\infty$ and $-\infty$; this results in the extended real line $\overline{\mathbb{R}}$.
Alexandroff Compactification, One-Point Compactification ("Alexandroff-Kompaktifizierung, Ein-Punkt-Kompatifizierung")	Alexandroff extension is a way to extend a noncompact topological space by adjoining a single point in such a way that the resulting space is compact. More precisely, let X be a topological space. Then the Alexandroff extension of X is a certain compact space $\tilde{X}$ together with an open embedding $c: X \to \tilde{X}$ such that the complement of X in $\tilde{X}$ consists of a single point, typically denoted ω or ∞. The map c is a Hausdorff compactification if and only if X is a locally compact, noncompact Hausdorff space. For such spaces the Alexandroff extension is called the **one-point compactification** or **Alexandroff compactification**. $\tilde{X} = X \cup \{\omega\}, \tilde{\mathcal{T}} = \mathcal{T} \cup \{S \cup \{\omega\}: X \backslash S \text{ is compact in } X\}$. <u>Example:</u> The 1-point compactification of $\mathbb{R}^n$ is homeomorpic to the n-dimensional sphere $S^n \subset \mathbb{R}^n$.
Equivalence relation ("Äquivalenzrelation")	An equivalence relation $\sim$ over a set X is a binary relation that is at the same time a reflexive relation, a symmetric relation and a transitive relation: (1) $x \sim x$ (reflexivity), (2) $x \sim y \Leftrightarrow y \sim x$ (symmetry), and (3) $x \sim y \land y \sim z \Longrightarrow x \sim z$ (transitivity)
Quotient Space ("Quotiententopologie")	Let $(X, \mathcal{T}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. The **quotient space**, $Y = X/\!\sim$ is defined to be the set of equivalence classes of elements of X: $Y = \{[x]: x \in X\} = \big\{\{v \in X: v \sim x\}: x \in X\big\}$ equipped with the topology $\mathcal{T}_Y$ where the open sets are defined to be those sets of equivalence classes whose unions are open sets in X.

Real Projective Space $\mathbb{RP}^n$ ("Reell-Projektiver Raum")	The **Real Projective Space** $\mathbb{RP}^n$ of dimension n is the topological space of lines passing through the origin $\vec{0}$ in $\mathbb{R}^{n+1}$. It is a compact, smooth manifold of dimension n. As with all projective spaces, $\mathbb{RP}^n$ is formed by taking the quotient of $\mathbb{R}^{n+1}\backslash\{\vec{0}\}$ under the equivalence relation $x \sim \lambda x$ for all real numbers $\lambda \neq 0$. For all x in $\mathbb{R}^{n+1}\backslash\{\vec{0}\}$ one can always find a λ such that λx has norm 1. There are precisely two such λ differing by sign. - Definition of the equivalence relation in $\mathbb{R}^{n+1}\backslash\{\vec{0}\}$ by $(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ for $\lambda \in \mathbb{R}\backslash\{0\}$. So, under this definition $\vec{x}\sim\vec{y} \Leftrightarrow \exists\lambda \neq 0: \vec{x} = \lambda\vec{y}$. This means: If the coordinates of a point are multiplied by a non-zero scalar then the resulting coordinates represent the same point ("homogeneous coordinates", see next point). - $\mathbb{RP}^n$ is the set of equivalence classes under $\sim$ denoted by $(x_0\!:x_1\!:\dots:x_n)$ (homogeneuous coordinates) - Every class has precisely two representatives with $x_0^2 + x_1^2 + \dots + x_n^2 = 1$ - In every $U_i \subseteq \mathbb{RP}^n$, determined by $x_i \neq 0$, one can choose a unique representative by $(x_0, x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots) \Longrightarrow each\ of\ U_i \leftrightarrow \mathbb{R}^n$. In other words: The set U_i that can be represented by homogeneous coordinates with $x_i = 1$ for some $i \geq 0$ form a subspace that be identified with$\mathbb{R}^n$. - As an <u>example</u>, take $\mathbb{R}^3$. In homogeneous coordinates, any point $(x\!:y\!:z)$ with $z \neq 0$ is equivalent to $(x/z\!:y/z\!:1)$. So there are two disjoint subsets of the projective plane: that consisting of the points $(x\!:y\!:z)$ =$(x/z\!:y/z\!:1)$ for $z \neq 0$, and that consisting of the remaining points $(x\!:y\!:0)$. The latter set can be subdivided similarly into two disjoint subsets, with points $(x/z\!:1\!:0)$ and $(x\!:0\!:0)$. This last point is equivalent to $(1\!:0\!:0)$. - This shows that $\mathbb{RP}^n$ can be covered by $n+1$ coordinate patches U_i that are isomorphic to $\mathbb{R}^n$. - Each patch $\mathbb{RP}^n\backslash U_i$ is isomorphic to $\mathbb{RP}^{n-1}$: $\mathbb{RP}^n\backslash U_i = \{(x_0, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)\} \leftrightarrow \mathbb{RP}^{n-1}$ - Projective space $\mathbb{RP}^n$ is therefore a disjoint union $\mathbb{RP}^n = \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \dots \cup \mathbb{R}^1 \cup \mathbb{R}^0$ (where $\mathbb{R}^0$ is a single point)
Disconnected ("unzusammenhängend")	A topological space $(X, \mathcal{T})$ is called **disconnected** if it is the union of two disjoint nonempty open sets. More formally, X is disconnected, if $X = \mathcal{O}_1 \cup \mathcal{O}_2$ for some open sets $\mathcal{O}_1 \neq \emptyset$ and $\mathcal{O}_2 \neq \emptyset$ with $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. <u>Remark:</u> obviously $\mathcal{O}_1 = X\backslash\mathcal{O}_2$ and $\mathcal{O}_2 = X\backslash\mathcal{O}_1$. These are also closed, so we could have made this definition also with closed set.
Connected ("zusammenhängend")	A topological space $(X, \mathcal{T})$ is called **connected** if it is not disconnected.
Path ("Weg")	A **path** in a topological space $(X, \mathcal{T})$ is a continuous map (i.e. function) f from the unit interval $I = [0,1]$ to X: More formally: Let $(X, \mathcal{T})$ be a toplogical space. Path $f = \{f: [0,1] \mapsto X: a, b \in X, \mathrm{f}(0) = a, \mathrm{f}(1) = b\}$
Pathwise Connected ("wegzusammenhänged")	A topological space $(X, \mathcal{T})$ is **pathwise connected** if for any two points $a \in X$, $b \in X$ there exists a path from a to b: $\forall a, b \in X\ \exists f: [0,1] \to X: f\ continous, \mathrm{f}(0) = a, \mathrm{f}(1) = b$. $pathwise\ connected \ \Longrightarrow\ connected$ (but not the other way!) <u>Counterexample</u> (connectd, but not pathwise connected): Consider the graph A of $y = \sin\left(\frac{1}{x}\right)\ over\ \mathbb{R}^+$ (subset of $\mathbb{R}^2$ under open topology of $\mathbb{R}^2$) with closure $\overline{A} = A \cup (\{0\} \times [-1,1])$. $\overline{A}$ is connected, but there is no path from the boundary $\overline{A} \cap A\ to\ A$
Loop ("Schleife")	A **loop** in a topological space X is a continuous function f from the unit interval $I = [0,1] \to X$ such that $\mathrm{f}(0) = \mathrm{f}(1)$. In other words, it is a path whose initial point is equal to the terminal point.
Invariance of connected components under homeomorphism	The number of connected components is invariant under homeomorphism (i.e. under a bijective map f such that both f and f^{-1} are continuous). Connectedness is therefore a topological invariant, i.e. a property that is invariant under homeomorphisms.
Homotopy ("Homotopie")	Two continuous maps $f: X \to Y, g: X \to Y$ are **homotopic** if there exists a continuous (meta-)map (a "map of maps") $F: X \times [0,1] \to Y$ with Euclidean product topology $F(x, 0) = \mathrm{f}(x), F(x, 1) = \mathrm{g}(x)\ \forall x \in X$. Homotopy is an equivalence relation.
Group ("Gruppe")	Informally, a group captures the essence of symmetry. The collection of symmetries of any object is a group, and every group is the symmetries of some object. Formally, a **group** is a set, G, together with an operation • (called the group law of G) that combines any two elements a and b to form another element, denoted $a \bullet b$ or ab. To qualify as a group, the set and operation, $(G, \bullet)$ must satisfy four requirements known as the group axioms: (1) For all $a, b \in G$, the result of the operation, $a \bullet b$, is also in G (closure), (2) for all $a, b, c \in G$, $(a \bullet b) \bullet c\ = a \bullet (b \bullet c)$ (associativity), (3) there exists an unique element $e \in G$ such that, $\forall a \in G: e \bullet a = a \bullet e$ (identity element), and (4) for each $a \in G$, there exists an element $b \in G$, such that $a \bullet b =\ b \bullet a = e$, (associativity). **Example:** Set of integers $\mathbb{Z}$. (1) For any two integers $a, b \in \mathbb{Z}$, the sum $(a+b)$ is also integer (2) for all integers $a, b, c \in \mathbb{Z}$: $(a+b)+c\ = a+(b+c)$ is true; (3) if $a \in \mathbb{Z}$, then $0 + a = a +\ 0\ =\ a$ (with 0 being the identity element); and (4) for every integer a, there is an integer b such that $a + b = b + a = 0$. The integer b is called the inverse element of the integer a.
Abelian Group ("Abelsche Gruppe")	An **Abelian Group** A is a group that in addition to the four group axioms also satisfies commutativity: $\forall a, b \in A: a \bullet b =\ b \bullet a$. Example: Set of integers $\mathbb{Z}$ with the operation addition "+".
Fundamental Group π_1 ("Fundamentalgruppe π_1")	The **fundamental group** $\pi_1(Y)$ is the set of all homotopic classes f from a circle to Y. <u>Group structure:</u> Every $f: S^1 \to Y$ corresponds to a closed path $\mathrm{f}(0) = \mathrm{f}(1) = x_0$. Unit element: $f = x_0 = const.$ Composition: $\mathrm{f} \circ \mathrm{g}(t) = \begin{cases} \mathrm{f}(2t-1) \dots t \geq \frac{1}{2} \\ g(2t) \quad\ \dots t < \frac{1}{2} \end{cases}$ (f and g start and end at x_0) Inverse: $\mathrm{f}^{-1}(t) = \mathrm{f}(1-t)$. The group structure is independent of x_0 if Y is pathwise connected.
Theorem about $\pi_1(Y \times \tilde{Y})$	Let Y and $\tilde{Y}$ be topological spaces. Then the fundamental group of their product space $\pi_1(Y \times \tilde{Y}) = \pi_1(Y) \oplus \pi_1(\tilde{Y})$ where the direct product '$\oplus$' is defined by $G \oplus \tilde{G} = \{(g, \tilde{g}): g \in G, \tilde{g} \in \tilde{G}\}$ with the group structure $(g_1, \tilde{g}_1)(g_2, \tilde{g}_2) = (g_1 g_2, \tilde{g}_1 \tilde{g}_2)$.

Simply Connected	Let Y be a topological space. Y is called **simply connected** if it is pathwise connected and its fundamental group $\pi_1(Y) = e$, with e being the unit element.
Covering Space ("Überlagerung")	$(\tilde{X}, \tilde{\mathcal{T}})$ is called a **covering space** of $(X, \mathcal{T})$ if there exists a continuous surjective map $\pi: \tilde{X} \to X$ such that every $x \in X$ has a neighbourhood $\mathrm{U}(x)$ such that π is a homeomorphism from $\tilde{U}$ to $\mathrm{U}(x)$ for every connected component $\tilde{U}$ of $\pi^{-1}\,\mathrm{U}(x)$. Loosely speaking, $\tilde{X}$ locally looks like X.
Universal Cover (Universelle Überlagerung)	$\tilde{X}$ is called the **universal cover** if $\pi_1(\tilde{X})$ is trivial, i.e. if it exists and is unique up to a homeomorphism for well-behaved spaces.) The universal cover (of the space X) covers any connected cover (of the space X). $Universal\ cover\ =$ $\{classes\ of\ maps: \mathrm{f}[0{:}\,1] \to X:\ \mathrm{f}(0) = x_0, \mathrm{f} \sim \mathrm{g}\ if\ \ \mathrm{f}(1) = \mathrm{g}(1)\ \ and\ the\ loop\ determined\ by\ \mathrm{fg}^{-1}\ is\ trivial\}$

Manifolds and Homology *(Mannigfaltigkeit und Homologie)*

Manifold ("Mannigfaltigkeit")	A **manifold** M is a topological space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n. <u>Examples:</u> One-dimensional manifolds include lines and circles, but not figure eights (because they have crossing points that are not locally homeomorphic to Euclidean 1-space). Two-dimensional manifolds are also called surfaces. Examples include the plane, the sphere, and the torus, but also the Klein bottle and real projective plane.
Differentiable C^r Manifold (Diff.bare Mannigfaltigkeit)	A n-dimensional **differentiable $\boldsymbol{C^r}$-manifold** M (where r stands for r times differentiable) is a Haussdorf space with a C^r atlas (where $r = \{\infty, 0, 1, 2, \dots\}$.
C^r Atlas ("C^r Atlas")	A $\boldsymbol{C^r}$ **atlas** is a set of charts $(U_i, x_{(i)})$ where U_i are open subsets of $M$and the $x_{(i)}$ are condinous invertible (i.e. homeomorphic) maps of U_i to open subsets of $\mathbb{R}^n$ such that (1) all of M is covered by all U_i: $M = \bigcup_i U_i\,, i \in I$, and (2) $U_i \cap U_j \neq \emptyset \Longrightarrow x_{(i)}, x_{(j)}^{-1}$ is r times continuously differentiable on $x_{(j)}(U_i \cap U_j)$
Compatible atlases	Two **compatible atlases** (i.e. atlases witch charts obeying condition (2)) are understood to define the same manifold.
Analytic Manifold (Analytische Mannigfaltigk.)	**Analytic manifolds** (C^r replaced by 'analytic') are smooth manifolds with the additional condition that the transition maps are analytic (they can be expressed as power series).
Orientable Manifold ("Orientierbare Mannigf.")	Let M be a differentiable manifold. M is **orientable** if there exists an atlas $\{(U_i, x_{(i)})\}$ such that the Jacobian determinant det $\left(\frac{\partial\left(x_{(i)}^1,\dots,x_{(i)}^n\right)}{\partial\left(x_{(j)}^1,\dots,x_{(j)}^n\right)}\right)$ (where $x_{(i)}^n$ denotes the n^{th} variable an i the i^{th} coordinate in $\mathbb{R}^n$) is positive for all non-empty $U_i \cap U_j$.
Paracompact ("parakompakt")	A manifold M is **paracompact** if for every atlas $\{(U_i, x_{(i)})\}$ there exists an atlas $\{(V_i, y_{(i)})\}$ with neighborhood $V_j \subset U_i$ for some i, such that every point in M has a neighborhood intersecting only finitely many V_j.
Diffeomorphic ("diffeomorph")	The manifold M and M' (*speak: "M prime"*) are called **diffeomorphic** if $\exists f: M \to M'$ such that $x' f x^{-1}$ is C^r an invertible (with inverse also C^r) wherever it is defined with respect to charts (U, x), (U', x') respectively.
Lie Group ("Lie Gruppe")	Informally, a **Lie Group** is a group of symmetries where the symmetries are continuous. A circle has a continuous group of symmetries: you can rotate the circle an arbitrarily small amount and it looks the same. Formally, a Lie Group G is a (finite dimensional smooth) differentiable manifold that is at the same time a group such that the group multiplication $f: G \times G \to G$ with $\mathrm{f}(x, y) = xy^{-1}$ is differentiable.
Group Action ("Gruppenoperation")	Informally, a **group action** is a way of interpreting the manner in which the elements of the group correspond to transformations of some space in a way that preserves the structure of that space. Formally, a group action on a manifold is a differentiable map $\sigma: G \times M \to M$ such that $\sigma_g \circ \sigma_h = \sigma_{gh}$ (left group action ghx), or $\sigma_h \circ \sigma_g = \sigma_{hg}$ (right group action xhg), where $\sigma_g(x) = \sigma(g, x)$
Effective Group Action ("effektive Operation")	Informally, a group action is **effective** if every element, except for the unit element, does something. Formally, a group action is effective if only the identity element e acts trivially: $\sigma_g(x) = x\ \forall x \in M \Longrightarrow g = e$. <u>Example</u>: $M = \mathbb{R}^n, G = group\ of\ rotations$.
Free Group Action ("freie Operation")	A group action is **free** if only σ_e has fixed points: $\sigma_g(x) \neq x\ \forall x \in M, g \in G\backslash\{e\}$
Transitive Group Action ("transitive Operation")	A group action is **transitive** if "all points can be moved": $\forall x, y \in M \exists g \in G{:}\, y = \sigma_g(x)$
Isotropy Group ("Isotropiegruppe")	The **isotropy group** (also called little group or stabilizer) of a point $x \in M$ is the subgroup $\mathrm{H}(x) = \{g \in G{:}\, \sigma_g(x) = x\}$ of G consisting of all the group elements that have x as a fixed point.
Classical Lie Groups ("Klassische Lie-Gruppen")	**Classical Lie Groups** can be represented by matrices. Consider a vector space $V \cong \mathbb{F}^n$ (where $\cong$ means 'isomorphic' and $\mathbb{F}^n$ is a field ("Körper"), typically $\mathbb{R}^n$ or $\mathbb{C}^n$). Given a basis of V, any $f \in \mathrm{Aut}(V)$ is represented by an invertible matrix $M \in \mathrm{GL}(n, \mathbb{F})$ (where $\mathrm{Aut}(V)$ is an automorphism and GL stands for "general linear") • $\mathrm{SL}(n, \mathbb{F})$: Group of matrices with determinant 1 • $\mathrm{SO}(n, \mathbb{F})$: Group of orthogonal matrices with det=1. Orthogonal matrices leave the metric $g_{mn} = \delta_{mn}$ of the Euclidean space invariant. • $\mathrm{Sp}(2n, \mathbb{F})$: Group of $2n \times 2n$-matrices that leave the n-fold tensor product invariant.

Simplicial Homology ("Simpliziale Homologie")

Simplicial Homology ("Simpliziale Homologie")	**Simplicial homology** formalizes the idea of the number of holes of a given dimension in a simplicial complex. It provides a way to study topological spaces whose building blocks are n-simplices. By definition, such a space is homeomorphic to a simplicial complex by a triangulation of the given space.
Orientation ("Orientierung")	An **orientation** of a k-simplex is given by an ordering of the vertices, written as $(v_0, \dots, v_k)$, with the rule that two orderings define the same orientation if and only if they differ by an even permutation.
Affine Space ("Affiner Raum")	**Affine space** is a geometric structure that generalizes the properties of Euclidean spaces in such a way that these are independent of the concepts of distance and measure of angles, keeping only the properties related to parallelism and ratio of lengths for parallel line segments.
Barycentric Coordinates ("Baryzentr. Koordinaten")	Let $\vec{p}_1, \dots, \vec{p}_n$ be the vertices ("Eckpunkte") of a simplex in an affine space A. The vertices themselves have the coordinates $\vec{p}_1 = \{1,0,0,..,0\}, \vec{p}_2 = \{0,1,0,..,0\}, \dots, \vec{p}_n = \{0,0,0,..,1\}$. If, for some point $\vec{x}$ in A, $(c_1 + \cdots + c_n)\vec{x} = c_1 x_1 + \cdots + c_n x_n$ and at least one of $c_1 \dots c_n$ does not vanish then the coefficients $c_1 \dots c_n$ are **barycentric coordinates** of $\vec{x}$ with respect to $p_1, \dots, p_n$. Often the values of coordinates are restricted with a condition $\sum c_i = 1$, which makes them unique. Such coordinates are called **absolute barycentric coordinates.**
Convex Hull ("Konvexe Hülle")	The **convex hull** $\mathrm{Conv}(\vec{p}_0, \vec{p}_1, \dots, \vec{p}_k)$ of a set X of points $(\vec{p}_0, \vec{p}_1, \dots, \vec{p}_k)$ in an Euclidean space (or, more generally, in an affine space over the reals) is the smallest convex set that contains X. For instance, when X is a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around X.
Simplex ("Simplex")	A **simplex** is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Simplex $\sigma: (\vec{p}_0, \vec{p}_1, \dots, \vec{p}_k) \stackrel{\text{def}}{=} \{\vec{x} \in \mathbb{R}^n : x = \sum_{i=0}^{k} c_i \vec{p}_i, c_i \geq 0, \sum_{i=0}^{k} c_i = 0\} = \mathrm{Conv}(\vec{p}_0, \vec{p}_1, \dots, \vec{p}_k)$. If σ lies in a k-dimensional subspace of $\mathbb{R}^n \rightarrow \dim(k)$
Oriented simplex	$\sigma: (p_0, \dots, p_r)$ is oriented if $(p_0, \dots, p_r) = (-1)^{\pi}(p_{\pi(0)}, \dots, p_{\pi(r)})$ for π being a permutation of $\{0, \dots, r\}$
Face	The convex hull $\rho = \mathrm{Conv}(S) : S \subseteq \{\vec{p}_0, \vec{p}_1, \dots, \vec{p}_k\}$ of any m points of an k-simplex is also a simplex, called an $\boldsymbol{m}$**-face**. The 0-faces $(\dim(\rho) = 0)$ are called the *vertices* ("Eckpunkte"), the 1-faces $(\dim(\rho) = 1)$ are called the *edges* ("Kanten"), the $(k-1)$-faces $(\dim(\rho) = k-1)$ are called the *facets* ("Facetten"), and the sole k-face is the whole n-simplex itself. All m-faces with $m < k$ are called *proper faces*. The empty set and the sole k-face are called *improper faces.*
Simplicial Complex "Simplizialkomplex"	A **simplicial complex** $\boldsymbol{K}$ is a finite set K of simplices in $\mathbb{R}^n$ such that: • $\sigma \in K$ ("every face of σ is in K") • $\sigma_i, \sigma_j \in K \Longrightarrow \sigma_i \cap \sigma_j = \emptyset \vee \sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j.
Polyhedron of simplicial complex K	A **polyhedron of a simplicial complex** $\boldsymbol{K}$ is defined as $\bigcup_{\sigma_i \in K} \sigma_i$
Triangulation ("Triangulierung")	A **triangulation** of a topological space X is a simplicial complex K, homeomorphic to X, together with a homeomorphism $h : K \rightarrow X$. A topological space is trianguable if it is homeomorphic to a polyhedron of some simplicial complex. This is true for differentiable manifolds in 2D and 3D, but generally not for 4D.
Simplicial r-chain ("simpliziale r-Kette")	A **simplicial r-chain** is a finite sum $\sum_{i=1}^{N} c_i \sigma_i$ where each c_i is an integer and σ_i is an oriented k-simplex: $\{\sum_{i=1}^{N} c_i \sigma_i : c_i \in \mathbb{Z}, \sigma_i \in K, \dim(\sigma_i) = r\}$
r-Chain Group	The **r-chain group** $\mathrm{C}_r(k)$ is the abelian group freely generated by the r-simplices $\{\sum_{i=1}^{N} c_i \sigma_i : c_i \in \mathbb{Z}, \sigma_i \in K, \dim(\sigma_i) = r\}$
Boundary Operator ("Randabbildung")	Let $\sigma: (p_0, \dots, p_r)$ be an oriented r-simplex. The **boundary operator** $\partial_r: C_r \rightarrow C_{r-1}$ is the homomorphism defined by $\partial_r(\sigma) = \sum_{i=0}^{r} (-1)^i (p_0, \dots, \hat{p}_i, \dots, p_r)$ where $(p_0, \dots, \hat{p}_i, \dots, p_r)$ is the i[th] face of σ, obtained by deleting its i[th] vertex. $\partial_{r-1}(\partial_r(\sigma)) = \partial_{r-1} \circ \partial_r = 0$.
Cycle Group	The **cycle group** $Z_r = \ker(\partial_r)$
Boundary Group	The **boundary group** $B_{r-1} = \mathrm{im}(\partial_r)$; $\partial_{r-1} \circ \partial_r = 0 \Longrightarrow B_r \subseteq Z_r$
Simplicial Homology Group	The **simplicial homology groups** $\mathrm{H}_r(K)$ of a simplicial complex K are defined using the simplicial chain complex $\mathrm{C}(K)$, with $\mathrm{C}_r(K)$ the free abelian group generated by the r-simplices of K: $\mathrm{H}_r(K) = \mathrm{Z}_r(K) / \mathrm{B}_r(K)$. The most general form of $\mathrm{H}_r(K)$ *is* $\mathrm{H}_r(K) \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{f} \oplus \underbrace{\mathbb{Z}_{k1} \oplus \dots \oplus \mathbb{Z}_{kp}}_{p}$. The first f factors form a free Abelian group of $\mathrm{rank}\, f$ and the next p factors are called the torsion subgroup of $\mathrm{H}_r(K)$.
Betti Numbers ("Betti-Zahlen")	Informally, the r^{th} **Betti Number** refers to the number of r-dimensional holes on a topological surface. The first few Betti numbers have the following definitions for 0-dimensional, 1-dimensional, and 2-dimensional simplicial complexes: b_0 is the number of connected components, b_1 is the number of one-dimensional or "circular" holes, b_2 is the number of two-dimensional "voids" or "cavities". Formally, The r^{th} Betti number represents the rank of the r^{th} homology group, denoted H_r: $b_r = \dim H_r(K, \mathbb{R})$.
Euler Characteristic	The **Euler characteristic** (or Euler number, or Euler–Poincaré characteristic) is a topological invariant. It is a number that describes a topological space's shape or structure regardless of the way it is bent. This means that any two surfaces that are homeomorphic must have the same Euler characteristic. The Euler characteristic χ was classically defined for the surfaces of polyhedra, according to the formula $\chi = V - E + F$ where V, E, and F are respectively the numbers of vertices (corners), edges and faces in the given polyhedron. For example, for a Tetrahedron $\chi = V - E + F = 4 - 6 + 4 = 2$. Similar, for a simplicial complex, the Euler characteristic equals the alternating sum $\chi = I_0 - I_1 + I_2 - \cdots$ where I_r is the number of r-simplices in k. Hence, $\chi(K) = \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r$
Connected Sum ("Verbundene Summe")	A **connected sum** of two m-dimensional manifolds is a manifold formed by deleting an open ball from each manifold and gluing together the resulting boundary spheres. Let M_1 and M_2 be two smooth manifolds of equal dimension n. Then the connected sum is denoted $M_1 \# M_2$.
Euler Characteristic of connected sums	Let M_1 and M_2 be two smooth manifolds of equal dimension n. Then the **euler characteristic of the connected sum** $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$.

Connected manifolds	Examples: (1) $Cylinder \cong S^1 \times \mathbb{R}$, (2) $Möbius\ Strip \cong S^1 \tilde{\times} \mathbb{R}$ (with $\tilde{\times}$ being the twisted product). The Möbius strip is non-orientable and has only one boundary component. (3) Torus $T^2 \cong S^1 \times S^1$

Homology for Sub-Manifolds (“Homologie von Untermannigfaltigkeiten”)

Sub-Manifold (“Untermannigfaltigkeit”)	Given the manifolds M and N and an injective map $\mathrm{f}: M \to N$. If $\mathrm{f}(M)$ is diffeomorphic to M then $\mathrm{f}(M)$ is a **sub-manifold** of N.
Manifold with Boundary (“Berandete Mannikfalt.”)	A **manifold with boundary** is defined like an ordinary manifold, but allowing charts in $\mathbb{R}_+^n \stackrel{\text{def}}{=} \mathbb{R} \cap \{x \geq 0\}$. Signs of such manifolds are derived from some suitable triangulation. The definition of chains, boundaries, boundary operators on chains and betti-numbers remain unchanged.
Homologous Manifolds (“Homologe Mannigf.”))	Two manifolds are **homologous** if their difference is a boundary.
Intersection	Assuming a Manifold M is oriented, chains ρ_k, ρ_{n-k} intersect traversely at $p \in M$ if $\det\left(\frac{\partial(\rho_1,\ldots,\rho_k,t_1,\ldots,t_{n-k})}{\partial(x_1,\ldots,x_n)}\right) \neq 0$ for $\rho_x(t)$ oriented parametrizations of ρ_k, ρ_{n-k}, M.
Intersection number (“Schnittzahl”)	${}^{\#}(\rho_K \circ \rho_{N-K}) = \sum_{p\in\rho_k\cap\rho_{n-k}} \operatorname{sign}\left(\frac{\partial(\rho,t)}{\partial(x)}\right)$. Depends only on the homology class.
Poincaré Duality (“Poincaré Dualität”)	Any linear functional $H_{n-k} \to \mathbb{Z}$ can be expressed as intersection with some $\rho_k \in H_K$. ${}^{\#}(\rho_{n-k} \circ \rho_l) = 0 \forall \rho_k \in H_l \Longrightarrow \rho_{n-k}$ is a torsion class.
Genus (“Geschlecht”)	Every compact connected surface is of the form ${}^{\#g}T^2, g = \{0,1,2,\ldots\}: orientable, g = genus$. The genus g of a closed orientable surface is the “number of handles”, or (equally) the “number of holes”. The euler characteristics of a closed orientable surface calculates as $\chi = 2 - 2g$. The genus k of a closed non-orientable surface is the number of real projective planes in a connected sum decomposition of the surface. The Euler characteristic can be calculated as $\chi = 2 - k$.
Crosscap (“Kreuzhaube”)	The crosscap can be thought of as the object produced by removing a small open disc in a surface and then identifying opposite sides. That is equivalent to gluing a möbius strip into the hole and taking the connected sum with $\mathbb{RP}^2$
Attaching a handle (“Henkel ankleben”)	Cut out two discs, identify boundaries. The Euler characteristic of the surface resulting from S^2 by attaching h handles and c crosscaps has $\chi = 2 - 2h - c$.

Differential Aspects of Manifolds (“Differentialaspekte von Mannigfaltigkeiten”)

Tangent Space (“Tangentialraum”)	Informal description: To every point p of a differentiable manifold a tangent space can be attached. The tangent space is a real vector space that intuitively contains the possible directions in which one can tangentially pass point p. The elements of the tangent space at p are called the tangent vectors v_p at p. More formally, the **tangent space** $\mathrm{T}_p(M)$ of the differentiable manifold M (with $p \in M$) is the linear span (“lineare Hülle”) of the operators $\left.\frac{\partial}{\partial x^i}\right\rvert_p$ acting on functions that are differentiable in the neighborhood of p. $\hat{v}_p = {v_p}^i \frac{\partial}{\partial x^i}$ acts via $\hat{v}_p f = {v_p}^i \frac{\partial \mathrm{f}}{\partial x^i}$ (summation convention). Remark: Given a curve $C: x^i = x^i(t)$, then $\frac{\partial \mathrm{f}}{\partial t} = \frac{\partial \mathrm{f}}{\partial x^i}\frac{\partial x^i}{\partial t}$ is the “direction of C at point p” with $\frac{\partial x^i}{\partial t}$ being the velocity.
Tangents Space is a vector space	As $\hat{v}_p(\alpha f + \beta g) = \alpha\hat{v}_p f + \beta\hat{v}_p g$ and $\hat{v}_p(fg) = (\hat{v}_p f)g + f(\hat{v}_p g)$, tangent space is also a vector space.
Coordinate Transformation (“Koordinatentransform.”)	To simplify notation: ${v_p}^i \stackrel{\text{def}}{=} v^i$. Then $\hat{v} = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} \Longrightarrow \tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i} v^i$
Cotangent Space (“Kotangentialraum”)	The cotangent space $\mathrm{T}_p^*(M)$ is the dual space $\mathrm{Hom}(\mathrm{T}_p(M), \mathbb{R})$, dual to $\mathrm{T}_p(M)$ (Hom being the space of linear maps). The basis dual to $\left\{\frac{\partial}{\partial x^i}\right\}$ is denoted by $\{dx^{oi}\}$. $\langle dx^i, \frac{\partial}{\partial x^j}\rangle = {\delta^i}_j$. Cotangent vector: $\hat{u} = u_j\, dx^j = \tilde{u}_i\, d\tilde{x}^i$ $\Longrightarrow \tilde{u}_k = \frac{\partial x^j}{\partial \tilde{x}_k} u_j$
Tensor	A **tensor** T of type (k, l) is a map $T: \underbrace{T_p^* \times \ldots \times T_p^*}_{k\ times} \times \underbrace{T_p \times \ldots \times T_p}_{l\ times} \to \mathbb{R}$ that is linear in every argument. $\mathrm{T}(\underbrace{u^{(1)}, \ldots, u^{(k)}}_{covectors}, \underbrace{v_{(1)}, \ldots, v_{(l)}}_{vectors}) = \mathrm{T}\left(u_{i_1}^{(1)}\, dx^{i_1}, \ldots, v_{(l)}^{j_l} \frac{\partial}{\partial x^{j_l}}\right) = u_{i_1}^{(1)}, \ldots, v_{(l)}^{j_l}\, \mathrm{T}\left(dx^{i_1}, \ldots, \frac{\partial}{\partial x^{j_l}}\right) \stackrel{\text{def}}{=} {T^{i_1\ldots i_k}}_{j_1\ldots j_l}$
Tensor Transformation (“Tensortransformation”)	${\tilde{T}^{\tilde{\imath}_1\ldots\tilde{\imath}_k}}_{\tilde{\jmath}_1\ldots\tilde{\jmath}_l} = \frac{\partial \tilde{x}^{\tilde{\imath}_1}}{\partial x^{i_1}} \cdot \ldots \cdot \frac{\partial \tilde{x}^{\tilde{\imath}_k}}{\partial x^{i_k}} \cdot \frac{\partial x^{j_1}}{\partial \tilde{x}^{\tilde{\jmath}_1}} \cdot \ldots \cdot \frac{\partial x^{j_l}}{\partial \tilde{x}^{\tilde{\jmath}_l}} {T^{i_1\ldots i_k}}_{j_1\ldots j_l}$ with $i_1 \ldots i_k$ contravariant, and $j_1 \ldots j_l$ covariant indices.
Tensor Operations (“Operationen auf Tensoren”)	
Addition	Two tensors can only be added if they are of the same type: $T + S = {T^{i_1\ldots i_k}}_{j_1\ldots j_l} + {S^{i_1\ldots i_k}}_{j_1\ldots j_l}$
Contraction (“Kontraktion”)	$(k+1, l+1) \to (k, l)$: ${S^{i,i_1\ldots i_k}}_{j,j_1\ldots j_l} \to {T^{i_1\ldots i_k}}_{j_1\ldots j_l}$
Tensor Product (“Tensorprodukt”)	$(k,l),(k',l') \to (k+k', l+l')$: $T \otimes S\left(u^{(1)}, \ldots, u^{(k+k')}, v_{(1)}, \ldots, v_{(l+l')}\right) = T\left(u^{(1)}, \ldots, u^{(k)}, v_{(1)}, \ldots, v_{(l)}\right) S\left(u^{(k+1)}, \ldots, u^{(k+k')}, v_{(l+1)}, \ldots, v_{(l+l')}\right)$
Symmetrizer (“Symmetrisierer”)	$\mathrm{S}(\omega)\left(v_{(1)}, \ldots, v_{(l)}\right) = \frac{1}{l!}\sum_\pi \omega\left(v_{\pi(1)}, \ldots, v_{\pi(l)}\right)$ with π running over all permutations of $(1, \ldots, l)$
Anti-Symmetrizer (“Antisymmetrisierer”)	$\mathrm{A}(\omega)\left(v_{(1)}, \ldots, v_{(l)}\right) = \frac{1}{l!}\sum_\pi (-1)^\pi\, \omega\left(v_{\pi(1)}, \ldots, v_{\pi(l)}\right)$ with π running over all permutations of $(1, \ldots, l)$ and $(-1)^\pi = 1$ for even permutations, and $(-1)^\pi = -1$ for odd permutations. Notation: $\omega_{[ij]} = (\mathrm{A}(\omega))_{ij}$

Differential Form ("Differentialform")	A **differential form** of order p is a totally antisymmetric $(0,p)$-tensor so that $\omega = \mathrm{A}(\omega)$
Wedge Product ("äußeres Produkt")	The **wedge product** $\wedge$ of a p-form $\propto$ and a q-form β is defined as $\alpha \wedge \beta = \mathrm{f}(p,q)\,\mathrm{A}(\alpha \otimes \beta) \Longrightarrow$ $$(\alpha \wedge \beta)\big(v_{(1)}, \dots, v_{(p+q)}\big) = \frac{\mathrm{f}(p,q)}{(p+q)!}\sum_{\pi}(-1)^{\pi}\,\alpha\big(v_{\pi(1)}, \dots, v_{\pi(p)}\big)\cdot\beta\big(v_{\pi(p+1)}, \dots, v_{\pi(p+q)}\big)$$ $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \Longrightarrow \mathrm{f}(p+q,r)\,\mathrm{f}(p+q) = \mathrm{f}(p,q+r)\,\mathrm{f}(q,r) \Longrightarrow$ solved by $\mathrm{f}(p,q) = \frac{\mathrm{g}(p+q)}{\mathrm{g}(p)\,\mathrm{g}(q)}$ convention: $\mathrm{g}(p) \stackrel{\text{def}}{=} p! \Longrightarrow dx^{i_1} \wedge \dots \wedge dx^{i_p} = dx^{i_1} \otimes \dots \otimes dx^{i_p} \pm \text{permutations}(dx^{i_1} \otimes \dots \otimes dx^{i_p}) \Longrightarrow$ $(dx \wedge \mathrm{dy})\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1$ Alternative convention: $\mathrm{g}(p) \stackrel{\text{def}}{=} 1 \Longrightarrow dx^{i_1} \wedge \dots \wedge dx^{i_p} = \frac{1}{p!}(dx^{i_1} \otimes \dots \otimes dx^{i_p} \pm \text{perm.}(dx^{i_1} \otimes \dots \otimes dx^{i_p}))$ $\alpha \wedge \beta \wedge \gamma \wedge \delta = -\beta \wedge \alpha \wedge \gamma \wedge \delta = +\beta \wedge \gamma \wedge \alpha \wedge \delta = -\beta \wedge \gamma \wedge \delta \wedge \alpha = \cdots$
Exterior Derivative ("Äußere Ableitung")	The **exterior derivative** extends the concept of the differential of a function to differential forms of higher degree. It is the operator $d\colon \Lambda^p \to \Lambda^{p+1}$ (Λ being the space of p-forms on M, p the number of co-vectors) with the properties: (1) $d(\alpha + \beta) = d\alpha + d\beta$ (linearity); (2) $d^2 = 0$ (nilpotency); (3) on 0-forms (i.e. functions), $df = \frac{df}{\partial x_i} dx^i$; (4) $d(f\omega) = (df) \wedge \omega + f\,d\omega$ for f … function, ω…form (chain rule 1) Derived rule (chain rule 2): $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Lambda^p, \beta \in \Lambda^q$