Kreis- und Hyperbelfunktionen

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Wichtige Umformungen:

Merksatz: sin cos ◦ cos sin ◦ cos cos ◦ – sin sin	$\sin(-x) = -\sin x$; $\cos(-x) = \cos x$; $\tan(-x) = -\tan x$; $\cot(-x) = -\cot x$		
1. Summensatz:	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$	$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$
2. Summensatz:	$\sin(\alpha) + \sin(\beta) = 2\ \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin(\alpha) - \sin(\beta) = 2\ \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos(\alpha) + \cos(\beta) = 2\ \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos(\alpha) - \cos(\beta) = -2\ \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	
Doppelte Winkel:	$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = \frac{2\tan(\alpha)}{1+\tan^2(\alpha)}$ $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$	$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$	$\cot(2\alpha) = \frac{\cot^2(\alpha) - 1}{2\cot(\alpha)}$
Halbwinkelsatz:	$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$; $\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$	Produkt: $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$	
Produkte:	$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$	$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$	
Verkettungen:	$\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}$	$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$ $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$	$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$ $\tan(\arccos(x)) = \frac{\sqrt{1+x^2}}{x}$
Umformungen:	„Pythagoras im EH-Kreis“: $\sin^2\alpha + \cos^2\alpha = 1$	$\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1$	$a\sin(x) + b\cos(x) = \sqrt{a^2+b^2}\cos\left(x - \arctan\left(\frac{a}{b}\right)\right) =$ $\sqrt{a^2+b^2}\sin\left(x + \frac{\pi}{2} - \arctan\left(\frac{a}{b}\right)\right) = \sqrt{a^2+b^2}\,\mathrm{Im}\left(e^{i\left(\frac{\pi}{2} - \arctan\left(\frac{a}{b}\right)\right)}\right)$

$\sin(\pi - x) = \sin x$; $\cos(\pi - x) = -\cos x$; $\tan(\pi - x) = -\tan x$; $\cot(\pi - x) = -\cot x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$; $\cos\left(\frac{\pi}{2} - x\right) = \sin x$; $\tan\left(\frac{\pi}{2} - x\right) = \cot x$; $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\sin(x \pm \pi) = -\sin x$; $\cos(x \pm \pi) = -\cos x$; $\tan(x \pm \pi) = \tan x$; $\cot(x \pm \pi) = \cot x$

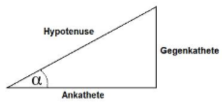
$\sin\left(x + \frac{\pi}{2}\right) = \cos x$; $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$; $\tan\left(x + \frac{\pi}{2}\right) = -\cot x$; $\cot\left(x + \frac{\pi}{2}\right) = -\tan x$

$\sin\left(x - \frac{\pi}{2}\right) = -\cos x$; $\cos\left(x - \frac{\pi}{2}\right) = \sin x$; $\tan\left(x - \frac{\pi}{2}\right) = -\cot x$; $\cot\left(x - \frac{\pi}{2}\right) = -\tan x$

α	0°	30°	45°	60°	90°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos\alpha$	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	0

Rechtwinkliges Dreieck:

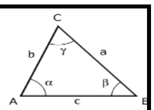
$$\sin\alpha = \frac{Gegenkathete}{Hypothenuse} \qquad \cos\alpha = \frac{Ankathete}{Hypothenuse} \qquad \tan\alpha = \frac{Gegenkathete}{Ankathete}$$



Schiefwinkliges Dreieck:

Sinussatz: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

Kosinusssatz:
$$a^2 + b^2 = c^2 + 2ab\cos\gamma$$
$$b^2 + c^2 = a^2 + 2bc\cos\alpha$$
$$c^2 + a^2 = b^2 + 2ca\cos\beta$$



Hyperbelfunktionen: Wichtige Umformungen:

„Pythagoras der EH-Hyp“:	$\cosh^2(x) - \sinh^2(x) = 1$	Erster und zweiter Summensatz analog zu Kreisfunktionen.	
Doppelte Winkel, etc:	$\sinh(2x) = 2\sinh(x)\cosh(x)$	$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$	$\cosh(x) + \sinh(x) = e^x$
	$a\cosh^2(x) + b\sinh^2(x) = a + (a+b)\sinh^2(x)$		

Zusammenhang zwischen Hyperbelfunktionen, Exponentialfunktion und Kreisfunktionen

$\boldsymbol{e^z}$	$e^z = \sum_{n=0}^{\infty} \frac{(\mathrm{Re}(z)+i\,\mathrm{Im}(z))^n}{n!} = e^{\mathrm{Re}(z)}(\cos(\mathrm{Im}(z)) + i\sin(\mathrm{Im}(z)))$	$\lvert e^z \rvert = e^{\mathrm{Re}(z)}$	$\arg(e^z) = \mathrm{Im}(z)$	$e^{z+2\pi k i} = e^z;\ k \in \mathbb{Z}$
Euler	$e^{\pm i\varphi} = \cos(\varphi) \pm i\sin(\varphi)$	**Moivre**	$(e^{i\varphi})^n = e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi)\,, n \in \mathbb{N}$	
$\mathbf{cos}(\boldsymbol{z})$	$\cos(z) = \cosh(iz) = \mathrm{Re}(e^{iz}) = \frac{e^{iz}+e^{-iz}}{2} = \cos(\mathrm{Re}(z))\cosh(\mathrm{Im}(z)) - i\sin(\mathrm{Re}(z))\sinh(\mathrm{Im}(z))$	$(\cos(z))^* = \cos(z^*)$		
$\mathbf{sin}(\boldsymbol{z})$	$\sin(z) = -i\sinh(iz) = \mathrm{Im}(e^{iz}) = \frac{e^{iz}-e^{-iz}}{2i} = \sin(\mathrm{Re}(z))\cosh(\mathrm{Im}(z)) + i\cos(\mathrm{Re}(z))\sinh(\mathrm{Im}(z))$	$(\sin(z))^* = \sin(z^*)$		
$\mathbf{tan}(\boldsymbol{z})$	$\tan(z) = -i\tanh(iz) = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$	$\mathbf{cot}(\boldsymbol{z})$	$\cot(z) = i\coth(iz) = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$	
$\mathbf{arcsin}(\boldsymbol{z})$	$\arcsin(z) = -i\ln\left(iz + \sqrt{1-z^2}\right)$	$\mathbf{arccos}(\boldsymbol{z})$	$\arccos(z) = \frac{\pi}{2} + i\ln\left(iz + \sqrt{1-z^2}\right)$	
$\mathbf{arctan}(\boldsymbol{z})$	$\arctan(z) = \frac{i}{2}(\ln(1-iz) - \ln(1+iz))$	$\mathbf{arccot}(\boldsymbol{z})$	$\mathrm{arcot}(z) = \frac{i}{2}\left(\ln\left(1-\frac{i}{z}\right) - \ln\left(1+\frac{i}{z}\right)\right)$	
$\mathbf{cosh}(\boldsymbol{z})$	$\cosh(z) = \cos(iz) = \frac{e^z+e^{-z}}{2} = \cosh(\mathrm{Re}(z))\cos(\mathrm{Im}(z)) + i\sinh(\mathrm{Re}(z))\sin(\mathrm{Im}(z))$			
$\mathbf{sinh}(\boldsymbol{z})$	$\sinh(z) = -i\sin(iz) = \frac{e^z-e^{-z}}{2} = \sinh(\mathrm{Re}(z))\cos(\mathrm{Im}(z)) + i\cosh(\mathrm{Re}(z))\sin(\mathrm{Im}(z))$			
$\mathbf{tanh}(\boldsymbol{z})$	$\tanh(z) = -i\tan(iz) = \frac{e^z-e^{-z}}{e^z+e^{-z}} = \frac{e^{2z}-1}{e^{2z}+1}$	$\mathbf{coth}(\boldsymbol{z})$	$\coth(z) = i\cot(iz) = \frac{e^z+e^{-z}}{e^z-e^{-z}}$	
$Ae^{ix} + Be^{-ix} = (A+B)\cos(x) + i(A-B)\sin(x)$		$Ae^x + Be^{-x} = (A+B)\cosh(x) + (A-B)\sinh(x)$		
$z = a + bi = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$	$\lvert z \rvert = \sqrt{a^2+b^2};\ \varphi = \arctan\frac{b}{a}$	$a = r\cos\varphi;\ b = r\sin\varphi$		

Sonstiges:

$x_1 = \arcsin(y)\,;\ x_2 = 180° - x_1;\ x_3 = 360° + x_1$	$x_1 = \arccos(y)\,;\ x_2 = 360° - x_1$
Goniometrisch Gleichung:	Kreisfrequenz aller Terme vereinheitlichen -> Alle Terme auf *eine* Funktion (z.B. sin) -> Subst. Krf = u
Anwendung 2. SS:	$\mathrm{krf}(\omega t + \varphi)\,\mathrm{krf}(\omega t) \rightarrow\ \frac{\alpha+\beta}{2} = \omega t + \varphi;\ \frac{\alpha-\beta}{2} = \omega t \rightarrow\ \alpha = 2\omega t + \varphi;\ \beta = \varphi \rightarrow\ \mathrm{krf}(2\omega t + \varphi) + \mathrm{krf}\,\varphi$