Lösung Beispiel 4,
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Translatorische kinetische Energie der Masse 1

In[30]:= $\text{TT1} = \frac{1}{2}\,\text{m1}\,\text{x1}'[t]^2;$

Translatorische kinetische Energie der Masse 2

In[31]:= $\text{TT2} = \frac{1}{2}\,\text{m2}\,\text{v2}[t]^2\ \text{/.}\ \{\text{v2}[t]^2 \to \text{D}[\text{x1}[t] + l\,\text{Sin}[\beta[t]], t]^2 + \text{D}[-l\,\text{Cos}[\beta[t]], t]^2\}$

Out[31]= $\frac{1}{2}\,\text{m2}\left(l^2\,\text{Sin}[\beta[t]]^2\,\beta'[t]^2 + \left(\text{x1}'[t] + l\,\text{Cos}[\beta[t]]\,\beta'[t]\right)^2\right)$

Rotatorische kinetische Energie der Masse 2

In[32]:= $\text{TR2} = \frac{1}{2}\,\text{II}\,\beta'[t]^2\ \text{/.}\ \{\text{II} \to \text{m2}\,l^2\}$

Out[32]= $\frac{1}{2}\,l^2\,\text{m2}\,\beta'[t]^2$

Potentielle Federenergie der Masse 1

In[33]:= $\text{U1} = \frac{1}{2}\,k\,\text{x1}[t]^2$

Out[33]= $\frac{1}{2}\,k\,\text{x1}[t]^2$

Potentielle Gravitationsenergie der Masse 2

In[34]:= $\text{U2} = \text{m2}\,g\,h\ \text{/.}\ \{h \to -l\,\text{Cos}[\beta[t]]\}$

Kosinus

Out[34]= $-g\,l\,\text{m2}\,\text{Cos}[\beta[t]]$

Gesamte kinetische Energie

In[35]:= $T = \text{TT1} + \text{TT2} + \text{TR2}$

Out[35]= $\frac{1}{2}\,\text{m1}\,\text{x1}'[t]^2 + \frac{1}{2}\,l^2\,\text{m2}\,\beta'[t]^2 + \frac{1}{2}\,\text{m2}\left(l^2\,\text{Sin}[\beta[t]]^2\,\beta'[t]^2 + \left(\text{x1}'[t] + l\,\text{Cos}[\beta[t]]\,\beta'[t]\right)^2\right)$

Gesamte potentielle Energie

In[36]:= $U = \text{U1} + \text{U2}$

Out[36]= $-g\,l\,\text{m2}\,\text{Cos}[\beta[t]] + \frac{1}{2}\,k\,\text{x1}[t]^2$

Lagrange Funktion:

In[37]:=
```
L = T - U
```

Out[37]=

$$g\, l\, m2 \,\mathrm{Cos}[\beta[t]] - \frac{1}{2} k\, x1[t]^2 + \frac{1}{2} m1\, x1'[t]^2 + \frac{1}{2} l^2 m2\, \beta'[t]^2 + \frac{1}{2} m2 \left(l^2 \mathrm{Sin}[\beta[t]]^2 \beta'[t]^2 + \left(x1'[t] + l\, \mathrm{Cos}[\beta[t]]\, \beta'[t]\right)^2\right)$$

Definiere konkrete Werte (u.A. m1=0,5 und m2=1)

In[38]:=
```
Werte1 = {g → 10, l → 5, k → 1, m1 → 0.5, m2 → 1};
```

Anfangsbedingungen

In[39]:=

$$AB = \left\{x1[0] == 0,\ \beta[0] == \frac{\pi}{4},\ x1'[0] == 0,\ \beta'[0] == 0\right\};$$

Definiere Lagrange-DGL

In[40]:=
```
DGL1 = D[D[L, x1'[t]], t] - D[L, x1[t]] == 0;
```
·· leite ab   leite ab
```
DGL2 = D[D[L, β'[t]], t] - D[L, β[t]] == 0;
```
·· leite ab   leite ab

Löse DGL numerisch

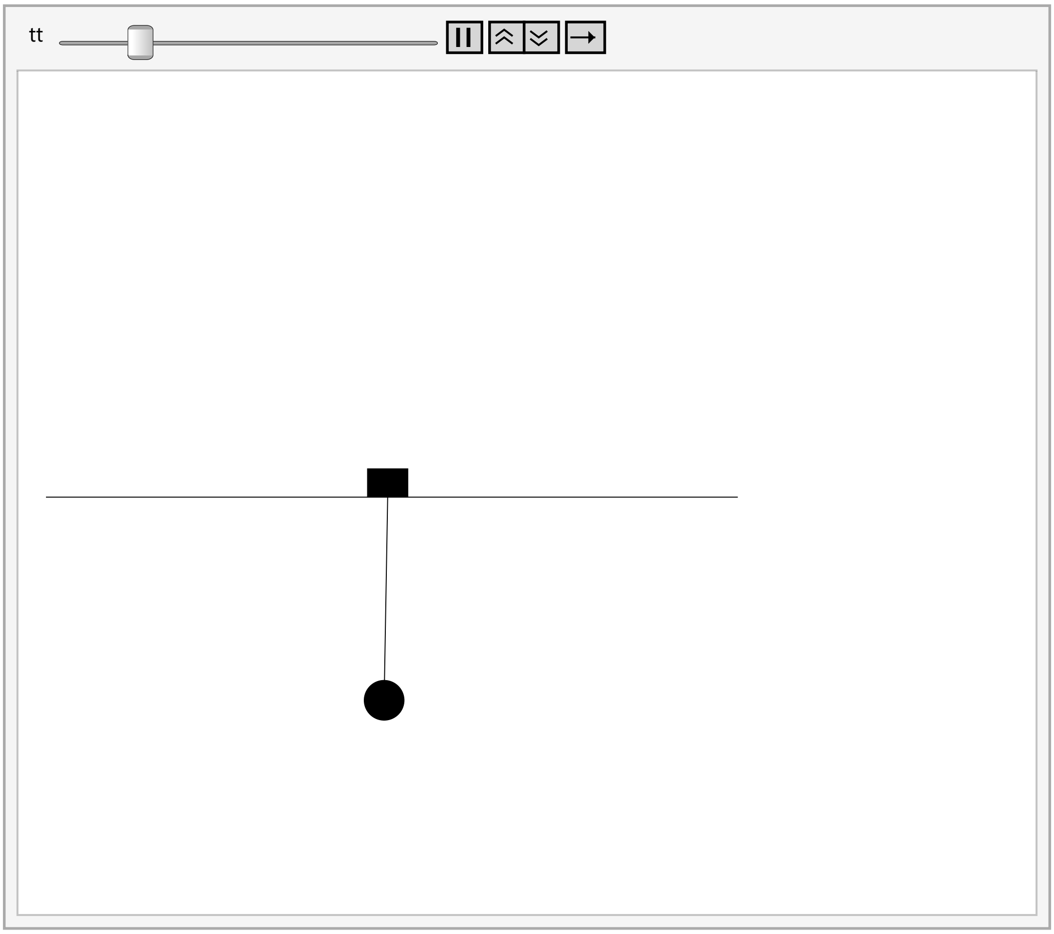
In[42]:=
```
Solution1 = NDSolve[{DGL1, DGL2, AB} //. Werte1, {x1[t], β[t]}, {t, 0, 200}]
```
löse Differentialgleichung numerisch

Out[42]=
```
{{x1[t] → InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ][t],
  β[t] → InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ][t]}}
```

Animation mit m1=0,5 und m2=1

In[43]:=
```
Animate[
  Graphics[
   {Line[{{-7, 0}, {10, 0}}],
    Line[{{x1[t] /. First[Solution1] /. t → tt, 0},
      {(x1[t] /. First[Solution1] /. t → tt) +
        l * Sin[β[t] /. First[Solution1] /. t → tt],
       -l * Cos[β[t] /. First[Solution1] /. t → tt]}}],
    Rectangle[{x1[t] - 0.5 /. First[Solution1] /. t → tt, 0.7},
     {x1[t] + 0.5 /. First[Solution1] /. t → tt, 0}],
    Disk[{(x1[t] /. First[Solution1] /. t → tt) +
        l * Sin[β[t] /. First[Solution1] /. t → tt],
       -l * Cos[β[t] /. First[Solution1] /. t → tt]}, 0.5`]},

   PlotRange → {{-7, 10}, {-7, 7}}],
  {tt, 0, 200}, AnimationRate → 3, AnimationRunning → False] /. Werte1
```

Out[43]=



Definiere neue Werte (u.A. m1=5 und m2=0.1)

```
In[44]:= Werte2 = {g → 10, l → 5, k → 1, m1 → 5, m2 → 0.1};
```

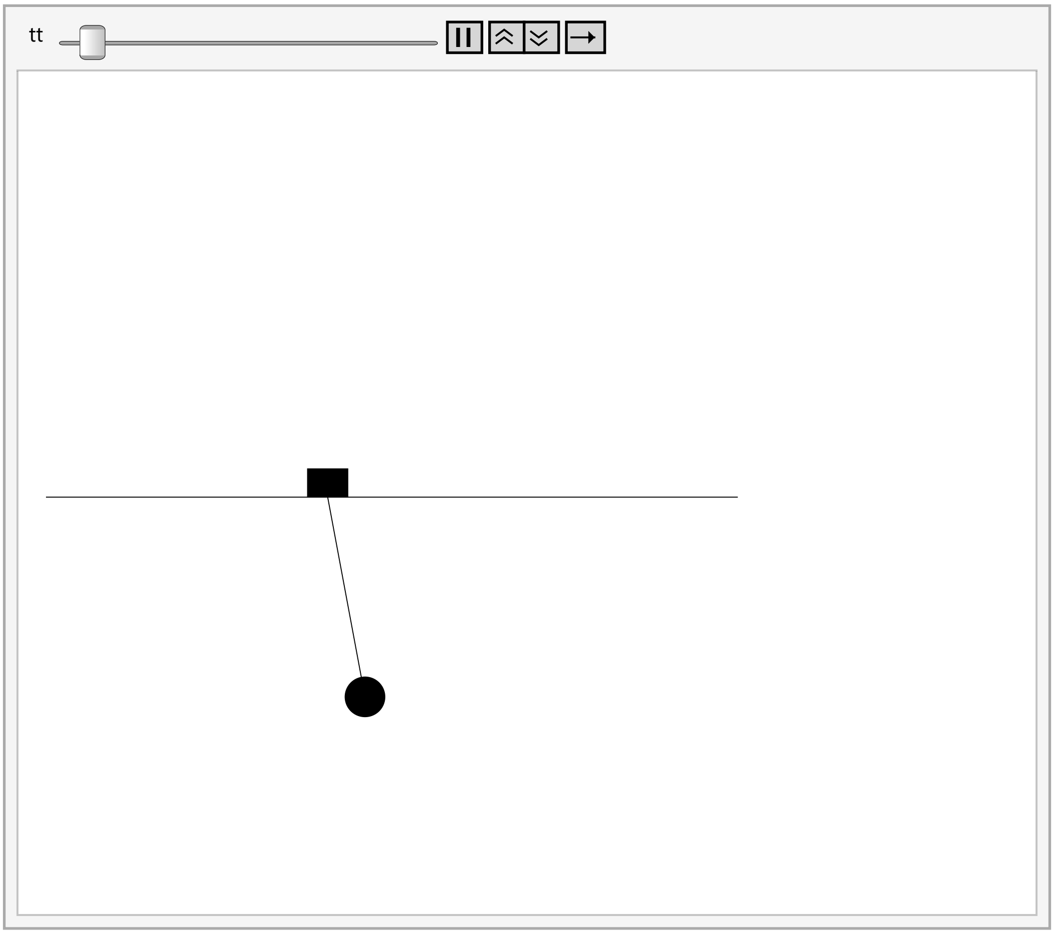
Löse DGL mit neuen Werten numerisch

```
In[45]:= Solution2 = NDSolve[{DGL1, DGL2, AB} //. Werte2, {x1[t], β[t]}, {t, 0, 200}]
                      löse Differentialgleichung numerisch

Out[45]= {{x1[t] → InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ][t],
   β[t] → InterpolatingFunction[ Domain: {{0., 200.}} Output: scalar ][t]}}
```

Animation mit m1=5 und m2=0,1

In[46]:=
```
Animate[
 Graphics[
  {Line[{{-7, 0}, {10, 0}}],
   Line[{{x1[t] /. First[Solution2] /. t → tt, 0},
     {(x1[t] /. First[Solution2] /. t → tt) +
       l * Sin[β[t] /. First[Solution2] /. t → tt],
      -l * Cos[β[t] /. First[Solution2] /. t → tt]}}],
   Rectangle[{x1[t] - 0.5 /. First[Solution2] /. t → tt, 0.7},
    {x1[t] + 0.5 /. First[Solution2] /. t → tt, 0}],
   Disk[{(x1[t] /. First[Solution2] /. t → tt) +
       l * Sin[β[t] /. First[Solution2] /. t → tt],
      -l * Cos[β[t] /. First[Solution2] /. t → tt]}, 0.5`]},

  PlotRange → {{-7, 10}, {-7, 7}}],
 {tt, 0, 200}, AnimationRate → 3, AnimationRunning → False] /. Werte2
```

Out[46]=



Linearisieren der DGL:

In[47]:=

```
DGL1LIN = FullSimplify[
          vereinfache vollständig
  Expand[DGL1 /. {Sin[β[t]] → β[t], Cos[β[t]] → 1} ] /. {β[t]^2 → 0, β'[t]^2 → 0} /.
  multipliziere aus  Sinus                Kosinus
   {β[t] → (x2[t] - x1[t])/l, β''[t] → D[(x2[t] - x1[t])/l, {t, 2}]}]
                                        leite ab

DGL2LIN = FullSimplify[Expand[DGL2 /. {Sin[β[t]] → β[t], Cos[β[t]] → 1} ] /.
          vereinfache voll··· multipliziere aus  Sinus          Kosinus
   {β[t]^2 → 0, β'[t]^2 → 0} /. {β[t] → (x2[t] - x1[t])/l,
   β''[t] → D[(x2[t] - x1[t])/l, {t, 2}]}, Assumptions → m2 > 0]
            leite ab                       Annahmen
```

Out[47]= $k\,x1[t] + m1\,x1''[t] + m2\,x2''[t] == 0$

Out[48]= $g\,x1[t] + l\,x1''[t] - 2\,l\,x2''[t] == g\,x2[t]$

Definiere neue Werte für linearisierte DGL

In[49]:=

```
Werte3 = {g → 9.8, l → 10, k → 0.2, m1 → 1, m2 → 2};
```

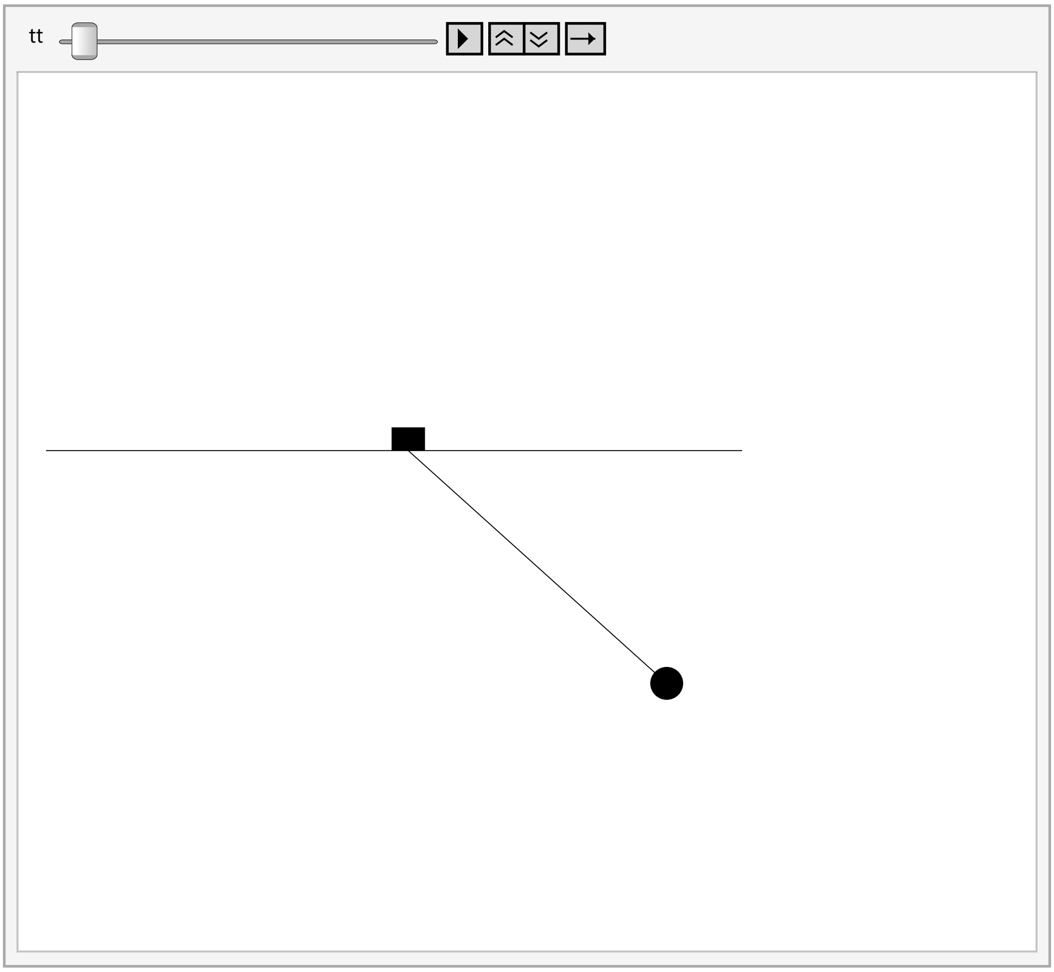
Exakte Lösung der linearisierten DGLs

```
In[50]:= Solution3 = DSolve[{DGL1LIN, DGL2LIN, AB /.
         {β[0] → (x2[t] - x1[t])/l /. t → 0, β'[0] → D[(x2[t] - x1[t])/l, t] /. t → 0}} //.
       Werte3, {x1[t], x2[t]}, t] // FullSimplify;
```

Animation mit linearisierten DGLs

```
In[51]:= Animate[
  Graphics[
   {Line[{{-11, 0}, {11, 0}}],
    Line[{{x1[t] /. First[Solution3] /. t → tt, 0},
      {x2[t] /. First[Solution3] /. t → tt, -l * Cos[
         1/l ((x2[t] /. First[Solution3]) - (x1[t] /. First[Solution3])) /. t → tt]}}],
    Rectangle[{x1[t] - 0.5 /. First[Solution3] /. t → tt, 0.7},
     {x1[t] + 0.5 /. First[Solution3] /. t → tt, 0}],
    Disk[{x2[t] /. First[Solution3] /. t → tt, -l * Cos[1/l ((x2[t] /. First[Solution3]) -
           (x1[t] /. First[Solution3])) /. t → tt]}, 0.5`]},
   PlotRange → {{-11, 10}, {-11, 7}}],
  {tt, 0, 200}, AnimationRate → 3, AnimationRunning → False] /. Werte3
```

Out[51]=



Eigenschwingungen:
Normalmodenansatz x1[t]=a Sin[ω t]; x2[t]=b Sin[ω t]

In[52]:=
```
gl1 = DGL1LIN /. {x1[t] → a Sin[ω t],
                              Sinus
    x1''[t] → -a ω^2 Sin[ω t], x2[t] → b Sin[ω t], x2''[t] → -b ω^2 Sin[ω t]}
                 Sinus                 Sinus                          Sinus
gl2 = DGL2LIN /. {x1[t] → a Sin[ω t], x1''[t] → -a ω^2 Sin[ω t],
                              Sinus                   Sinus
    x2[t] → b Sin[ω t], x2''[t] → -b ω^2 Sin[ω t]}
             Sinus                   Sinus
```

Out[52]= $a\, k \,\mathrm{Sin}[t\, \omega] - a\, m1\, \omega^2 \,\mathrm{Sin}[t\, \omega] - b\, m2\, \omega^2 \,\mathrm{Sin}[t\, \omega] == 0$

Out[53]= $a\, g \,\mathrm{Sin}[t\, \omega] - a\, l\, \omega^2 \,\mathrm{Sin}[t\, \omega] + 2\, b\, l\, \omega^2 \,\mathrm{Sin}[t\, \omega] == b\, g \,\mathrm{Sin}[t\, \omega]$

Bestimme Koeffizienten

In[54]:=
```
gl1 = gl1[[1]] / Sin[ω t] // FullSimplify // ExpandAll
gl2 = gl2[[1]] / Sin[ω t] - gl2[[2]] / Sin[ω t] // FullSimplify // ExpandAll
```
Sinus vereinfache vollstän·· multipliziere alle aus
Sinus Sinus vereinfache vollstä··· multipliziere a

Out[54]= $a\,k - a\,m1\,\omega^2 - b\,m2\,\omega^2$

Out[55]= $a\,g - b\,g - a\,l\,\omega^2 + 2\,b\,l\,\omega^2$

Bestimme Eigenmoden

In[56]:=
```
det = Solve[Det[( Coefficient[gl1, a]  Coefficient[gl1, b]
                  Coefficient[gl2, a]  Coefficient[gl2, b] )] == 0, ω]
```
löse Determinante

Out[56]=

$$\left\{\left\{\omega \to -\sqrt{\left(\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} - \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}\right)}\right\},\right.$$

$$\left\{\omega \to \sqrt{\left(\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} - \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}\right)}\right\},$$

$$\left\{\omega \to -\sqrt{\left(\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} + \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}\right)}\right\},$$

$$\left.\left\{\omega \to \sqrt{\left(\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} + \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}\right)}\right\}\right\}$$

(Nur positive Frequenzen berücksichtigen)

In[57]:=
```
ω1 = det[[2, 1, 2]]
ω2 = det[[4, 1, 2]]
```

Out[57]=
$$\sqrt{\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} - \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}}$$

Out[58]=
$$\sqrt{\frac{k\,l}{2\,l\,m1 + l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1 + l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1 + l\,m2)} + \frac{\sqrt{(-2\,k\,l - g\,m1 - g\,m2)^2 - 4\,g\,k\,(2\,l\,m1 + l\,m2)}}{2\,(2\,l\,m1 + l\,m2)}}$$

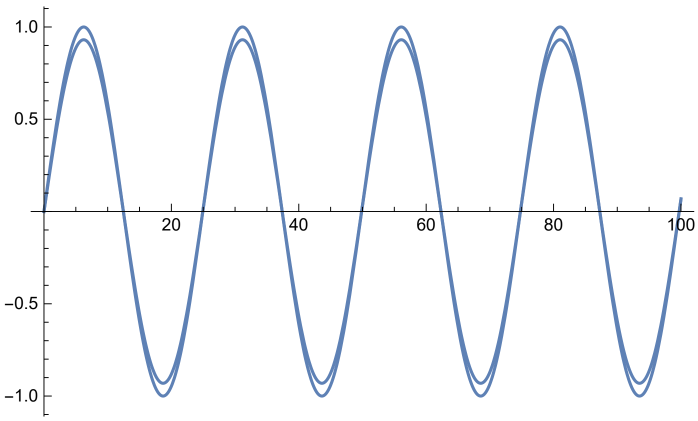
In[59]:=
```
a1 = Solve[{gl1 == 0, gl2 == 0} /. {ω → ω1, b → 1}, a][[1, 1, 2]]
```
löse
```
a2 = Solve[{gl1 == 0, gl2 == 0} /. {ω → ω2, b → 1}, a][[1, 1, 2]]
```
löse

Out[59]=
$$\frac{m2\left(\frac{k\,l}{2\,l\,m1+l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1+l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1+l\,m2)} - \frac{\sqrt{(-2\,k\,l-g\,m1-g\,m2)^2-4\,g\,k\,(2\,l\,m1+l\,m2)}}{2\,(2\,l\,m1+l\,m2)}\right)}{k - m1\left(\frac{k\,l}{2\,l\,m1+l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1+l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1+l\,m2)} - \frac{\sqrt{(-2\,k\,l-g\,m1-g\,m2)^2-4\,g\,k\,(2\,l\,m1+l\,m2)}}{2\,(2\,l\,m1+l\,m2)}\right)}$$

Out[60]=
$$\frac{m2\left(\frac{k\,l}{2\,l\,m1+l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1+l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1+l\,m2)} + \frac{\sqrt{(-2\,k\,l-g\,m1-g\,m2)^2-4\,g\,k\,(2\,l\,m1+l\,m2)}}{2\,(2\,l\,m1+l\,m2)}\right)}{k - m1\left(\frac{k\,l}{2\,l\,m1+l\,m2} + \frac{g\,m1}{2\,(2\,l\,m1+l\,m2)} + \frac{g\,m2}{2\,(2\,l\,m1+l\,m2)} + \frac{\sqrt{(-2\,k\,l-g\,m1-g\,m2)^2-4\,g\,k\,(2\,l\,m1+l\,m2)}}{2\,(2\,l\,m1+l\,m2)}\right)}$$

Grafische Darstellung

In[61]:=
```
Plot[{a1 Sin[ω1 t] , Sin[ω1 t]} /. Werte3, {t, 0, 100}]
```
stelle Fu··· Sinus Sinus

Out[61]=


In[62]:=
```
Plot[{a2 Sin[ω2 t] /. Werte3, Sin[ω2 t] /. Werte3}, {t, 0, 100}]
```
stelle Fu··· Sinus Sinus

Out[62]=