QED

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1. Starting Point: Schrödinger Equation

Schrödinger Equation:	$i\hbar\frac{\partial}{\partial t}\vert\Psi(t)\rangle = \hat{H}\vert\Psi(t)\rangle$	Hamiltonian:	$\hat{H} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r}) = \frac{\hat{p}^2}{2m} + V(\vec{r})$	Momentum Operator:	$\hat{\vec{p}} = -i\hbar\vec{\nabla}$	nonrelativistic because of $\hat{H}$ and fixed particle number
Time evolution operator	$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}} \Longrightarrow$ $\vert\Psi(t)\rangle = \hat{U}(t)\vert\Psi(0)\rangle$	Inserting into Schrödinger eq.	$i\hbar\frac{\partial}{\partial t}\left(e^{-\frac{i\hat{H}t}{\hbar}}\vert\Psi(0)\rangle\right) = \hat{H}\left(e^{-\frac{i\hat{H}t}{\hbar}}\vert\Psi(0)\rangle\right) \Longrightarrow i\hbar\left(-\frac{i\hat{H}}{\hbar}\right)e^{-\frac{i\hat{H}t}{\hbar}}\vert\Psi(0)\rangle = \hat{H}e^{-\frac{i\hat{H}t}{\hbar}}\vert\Psi(0)\rangle$			
SEQ nat. units	$i\partial_t\Psi = \hat{H}\Psi \Longrightarrow i\partial_t\Psi = \left(-\frac{1}{2m}\vec{\nabla}^2 + V\right)\Psi \Longrightarrow \partial_t\Psi = \left(i\frac{1}{2m}\vec{\nabla}^2 - iV\right)\Psi$ ... (1)					
conjugate:	$-i\partial_t\Psi^* = \left(-\frac{1}{2m}\vec{\nabla}^2 + V\right)\Psi^* \Longrightarrow \partial_t\Psi^* = \left(-i\frac{1}{2m}\vec{\nabla}^2 + iV\right)\Psi^*$ ... (2)					
Density	$\rho = \vert\Psi\vert^2 = \Psi^*\Psi \Longrightarrow \partial_t\rho = \partial_t(\Psi^*\Psi) = (\partial_t\Psi^*)\Psi + \Psi^*\partial_t\Psi \overset{(1)(2)}{\Longrightarrow} \partial_t\rho = \left(-i\frac{1}{2m}\vec{\nabla}^2 + iV\right)\Psi^*\Psi + \Psi^*\left(i\frac{1}{2m}\vec{\nabla}^2 - iV\right)\Psi \Longrightarrow$ $\partial_t\rho = -\frac{i}{2m}\vec{\nabla}^2\Psi^*\Psi + iV\Psi^*\Psi + \frac{i}{2m}\Psi^*\vec{\nabla}^2\Psi - iV\Psi^*\Psi = \frac{i}{2m}\left(\Psi^*\vec{\nabla}^2\Psi - (\vec{\nabla}^2\Psi^*)\Psi\right)\Big\vert + \vec{\nabla}\Psi^*\vec{\nabla}\Psi - \vec{\nabla}\Psi^*\vec{\nabla}\Psi$ $\partial_t\rho = \frac{i}{2m}\left(\vec{\nabla}\Psi^*\vec{\nabla}\Psi + \Psi^*\vec{\nabla}^2\Psi - (\vec{\nabla}^2\Psi^*)\Psi - \vec{\nabla}\Psi^*\vec{\nabla}\Psi\right) = \frac{i}{2m}\left(\vec{\nabla}\cdot(\Psi^*\vec{\nabla}\Psi) - \vec{\nabla}\cdot\left((\vec{\nabla}\Psi^*)\Psi\right)\right) \Longrightarrow$ $\partial_t\rho = \frac{i}{2m}\vec{\nabla}\cdot\left(\Psi^*\vec{\nabla}\Psi - (\vec{\nabla}\Psi^*)\Psi\right)$ ... (3). Because $\int_{-\infty}^{\infty}\vert\Psi\vert^2\,d^3x = 1 \Longrightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\vert\Psi\vert^2 d^3x = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\vert\Psi\vert^2 d^3x = \int_{-\infty}^{\infty}\partial_t\rho\, d^3x = 0.$ Check: $\int_{-\infty}^{\infty}\partial_t\rho\, d^3x \overset{(3)}{=} \frac{i}{2m}\int_{-\infty}^{\infty}\vec{\nabla}\cdot\left(\Psi^*\vec{\nabla}\Psi - (\vec{\nabla}\Psi^*)\Psi\right)d^3x = 0$ ■					
Pauli matrices	$\sigma_1 = \vert\uparrow\rangle\langle\downarrow\vert + \vert\downarrow\rangle\langle\uparrow\vert = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{\{S_z\}}$; $\sigma_2 = i(-\vert\uparrow\rangle\langle\downarrow\vert + \vert\downarrow\rangle\langle\uparrow\vert) = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^{\{S_z\}}$; $\sigma_3 = \vert\uparrow\rangle\langle\uparrow\vert - \vert\downarrow\rangle\langle\downarrow\vert = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\{S_z\}}$ $\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \mathbb{1}_2,\ \vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)^T,\ \vec{\sigma}^2 = 3\mathbb{1}_2,\ \left(\vec{\sigma}\cdot\hat{\vec{x}}\right)^2 = \mathbb{1}_2$ with $\hat{\vec{x}} \overset{\text{def}}{=} \frac{\vec{x}}{\vert\vec{x}\vert},\ \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$					

1.1. Relativistic Notation

Four-Vector	$x^\mu = \begin{pmatrix}ct\\ \vec{x}\end{pmatrix}$	Minkowski Metric	$g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ invariant under Lorentz Trafo $g^{\mu\nu}L^\mu{}_\sigma L^\nu{}_\rho = \sigma\rho$	Line element	$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$	Euclidian metric	$\delta_{ij} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	invariant under 3D rotation	
Lorentz-transformation:	Let S' be the „moving" system, and S the „rest" system; i.e. velocity of S' in relation to S determines value and sign of β.	**Active LT:** How does „moving" S' looks like in „rest" system S? $\boldsymbol{a^\mu = \Lambda^\mu{}_\nu a'^\nu}$	**Passive LT:** How does „rest" system S look like in moving" S'? $\boldsymbol{a'^\mu = \tilde{\Lambda}^\mu{}_\nu a^\nu}$						
Active LT Boost in x, $S' \to S$:	$L^\mu{}_\nu = \begin{bmatrix}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$	Passive LT Boost in x, $S \to S'$:	$\tilde{L}^\mu{}_\nu = \begin{bmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$	Four gradient	$\partial_\mu = \begin{pmatrix}\frac{1}{c}\partial_t\\ \vec{\nabla}\end{pmatrix};\ \partial^\mu = \begin{pmatrix}\frac{1}{c}\partial_t\\ -\vec{\nabla}\end{pmatrix}$	Quabla	$\square = \partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2$		
Four momentum	$p^\mu = m_0 u^\mu = m_0\begin{pmatrix}\gamma c\\ \gamma\vec{v}\end{pmatrix} = \begin{pmatrix}\frac{E}{c}\\ \vec{p}\end{pmatrix} = \begin{pmatrix}m_0 c + \frac{E_{kin}}{c}\\ \vec{p}\end{pmatrix}$	$p_\mu = \begin{pmatrix}\frac{E}{c}\\ -\vec{p}\end{pmatrix}$	$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m_0^2c^2$ ...invar.	$\hat{p}^\mu = i\hbar\partial^\mu = \begin{pmatrix}\frac{i\hbar}{c}\partial_t\\ -i\hbar\vec{\nabla}\end{pmatrix};\ \hat{p}_\mu = \begin{pmatrix}\frac{i\hbar}{c}\partial_t\\ i\hbar\vec{\nabla}\end{pmatrix}$					

1.2. Natural Units

$\hbar \overset{\text{def}}{=} 1$ ... combines energy and time	$c \overset{\text{def}}{=} 1$ ... combines length and time	$1\frac{1}{eV} \approx 200\text{nm}$	Fine structure const.	$\alpha = \frac{1}{4\pi\hbar c} \approx \frac{1}{137}$ (low energy)

2. Relativistic Wave Equation: Klein Gordon Equation

Naive approach	$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{p^2 + m^2}$ (ignoring the negative solutions) $\Longrightarrow \hat{H} = \sqrt{\hat{p}^2 + m^2} = m + \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3} + \cdots.$ **Problems:** (1) $\Longrightarrow$ time-space-asymmetric, (2) $\Longrightarrow$ contains derivation of arbitrary high order $\Longrightarrow$ non-local!
Klein-Gordon equation	$E^2 = p^2 + m^2 \Longrightarrow \hat{H}^2 = \hat{p}^2 + m^2 = \left(-i\vec{\nabla}\right)^2 + m^2 = -\vec{\nabla}^2 + m^2$ ... (1) Schrödinger: $i\partial_t\Psi = \hat{H}\Psi\vert i\partial_t\cdot \Longrightarrow -\partial_t^2\Psi = i\partial_t(\hat{H}\Psi) \Longrightarrow$ $-\partial_t^2\Psi = i(\partial_t\hat{H})\Psi + i\hat{H}\partial_t\Psi\vert\partial_t\hat{H} = 0 \Longrightarrow -\partial_t^2\Psi = \hat{H}i\partial_t\Psi\vert i\partial_t\Psi = \hat{H}\Psi \Longrightarrow -\partial_t^2\Psi = \hat{H}^2\Psi \overset{(1)}{\Longrightarrow} -\partial_t^2\Psi = \left(-\vec{\nabla}^2 + m^2\right)\Psi \Longrightarrow$ $-\partial_t^2\Psi = -\vec{\nabla}^2\Psi + m^2\Psi \Longrightarrow \partial_t^2\Psi - \vec{\nabla}^2\Psi + m^2\Psi = 0 \Longrightarrow \left(\partial_t^2 - \vec{\nabla}^2\right)\Psi + m^2\Psi = 0 \Longrightarrow \square\Psi + m^2\Psi = 0 \Longrightarrow \boxed{(\square + m^2)\Psi = 0}$
Density	$\square\Psi + m^2\Psi = 0 \Longrightarrow \square\Psi = -m^2\Psi\vert\Psi^*\cdot \Longrightarrow \Psi^*\square\Psi = -m^2\Psi^*\Psi$ ... (2) $\vert conj. \Longrightarrow \Psi\square\Psi^* = -m^2\Psi^*\Psi$ ... (3) (2) − (3) $\Longrightarrow \Psi^*\square\Psi - \Psi\square\Psi^* = 0 \Longrightarrow \Psi^*\partial_\mu\partial^\mu\Psi - \Psi\partial_\mu\partial^\mu\Psi^* = 0\Big\vert + \partial_\mu\Psi^*\partial^\mu\Psi - \partial_\mu\Psi^*\partial^\mu\Psi \Longrightarrow$ $\partial_\mu\Psi^*\partial^\mu\Psi + \Psi^*\partial_\mu\partial^\mu\Psi - \partial_\mu^{\uparrow}\Psi^*\partial_{\downarrow}^\mu\partial^\mu\Psi - \Psi\partial_\mu\partial^\mu\Psi^* = 0 \Longrightarrow \partial_\mu(\Psi^*\partial^\mu\Psi - \Psi\partial^\mu\Psi^*) = 0$ ... (4) (4) $\Longrightarrow \partial_0(\Psi^*\partial^0\Psi - \Psi\partial^0\Psi^*) = 0\vert\partial_0 = \partial^0 = \partial_t \Longrightarrow \partial_t(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*) = 0$ ... (5) (4) $\Longrightarrow \partial_i(\Psi^*\partial^i\Psi - \Psi\partial^i\Psi^*) = 0\vert\partial_i = \vec{\nabla}, \partial^i = -\vec{\nabla} \Longrightarrow \vec{\nabla}\left(-\Psi^*\vec{\nabla}\Psi + \Psi\vec{\nabla}\Psi^*\right) = 0 \Longrightarrow -\vec{\nabla}\cdot\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) = 0$ ... (6) $0 = 0 \overset{(5)(6)}{\Longrightarrow} \partial_t(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*) = -\vec{\nabla}\cdot\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right)\Big\vert vgl. \frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{j} \Longrightarrow \boxed{\rho \propto \Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*}$ ... not positive definite!

2.1 Dirac Equation

Start like Klein Gord	$E^2=p^2+m^2 \Longrightarrow \hat{H}_D^2=\hat{p}^2+m^2=\left(-i\vec{\nabla}\right)^2+m^2=-\vec{\nabla}^2+m^2$ ... (1) Schrödinger: $i\partial_t\Psi=\hat{H}_D\Psi \mid i\partial_t\cdot \Longrightarrow -\partial_t^2\Psi=i\partial_t(\hat{H}_D\Psi) \Rightarrow$ $-\partial_t^2\Psi=i(\partial_t\hat{H}_D)\Psi+i\hat{H}_D\partial_t\Psi \mid \partial_t\hat{H}_D=0 \Longrightarrow -\partial_t^2\Psi=\hat{H}_D i\partial_t\Psi \mid i\partial_t\Psi=\hat{H}_D\Psi \Longrightarrow -\partial_t^2\Psi=\hat{H}_D^2\Psi \overset{(1)}{\Rightarrow} -\partial_t^2\Psi=\left(-\vec{\nabla}^2+m^2\right)\Psi$ ... (2)
Ansatz	$\Psi(\vec{r},t)=\begin{pmatrix}\Psi_1(\vec{r},t)\\ \Psi_2(\vec{r},t)\\ \Psi_3(\vec{r},t)\\ \Psi_4(\vec{r},t)\end{pmatrix}$ (spinor $\in \mathbb{C}^4$) In order to allow Lorentz-covariance, $\hat{H}_D$, in x-space, must be linear in spatial derivatives: $\hat{H}_D=\underline{\alpha}_i\hat{p}_i+\underline{\beta}m$ ... (3) with $\underline{\alpha}_i$ and $\underline{\beta}$ being 4x4 matrices acting on the components of Ψ
Derivation of Dirac Equation	$(2)\Longrightarrow \hat{H}_D^2=-\vec{\nabla}^2+m^2 \Longrightarrow \hat{H}_D^2=-\partial_i\partial_i+m^2 \Longrightarrow \hat{H}_D^2=-\partial_i\partial_j\delta_{ij}+m^2 \overset{(3)}{\Rightarrow} \left(\underline{\alpha}_i\hat{p}_i+\underline{\beta}m\right)\left(\underline{\alpha}_j\hat{p}_j+\underline{\beta}m\right)=-\partial_i\partial_j\delta_{ij}+m^2 \Longrightarrow$ $\frac{1}{i}\underline{\alpha}_i\partial_i\frac{1}{i}\underline{\alpha}_j\partial_j+\underline{\beta}m\underline{\beta}m+\underline{\beta}m\underbrace{\frac{1}{i}\underline{\alpha}_j\partial_j}_{j\to i}+\frac{1}{i}\underline{\alpha}_i\partial_i\underline{\beta}m=\left(-\partial_i\partial_j\delta_{ij}+m^2\right)\mathbb{1}$ $-\underline{\alpha}_i\underline{\alpha}_j\partial_i\partial_j+m^2\underline{\beta}^2+\frac{1}{i}m\underline{\beta}\underline{\alpha}_i\partial_i+\frac{1}{i}m\underline{\alpha}_i\underline{\beta}\partial_i=\left(-\partial_i\partial_j\delta_{ij}+m^2\right)\mathbb{1}$ $-\underline{\alpha}_i\underline{\alpha}_j\partial_i\partial_j+m^2\underline{\beta}^2-im\left(\underline{\beta}\underline{\alpha}_i+\underline{\alpha}_i\underline{\beta}\right)\partial_i=\left(-\partial_i\partial_j\delta_{ij}+m^2\right)\mathbb{1}$ Coefficients of $-\partial_i\partial_j$: $\underline{\alpha}_i\underline{\alpha}_j=\delta_{ij}\mathbb{1}\begin{cases}\overset{if\ i=j}{\Longrightarrow}\underline{\alpha}_i\underline{\alpha}_i=\frac{1}{2}\left[\underline{\alpha}_i,\underline{\alpha}_i\right]_+=\mathbb{1}\\ \overset{if\ i\neq j}{\Longrightarrow}\underline{\alpha}_i\underline{\alpha}_j=\mathbb{0}=\underline{\alpha}_j\underline{\alpha}_i\Rightarrow\frac{1}{2}\left[\underline{\alpha}_i,\underline{\alpha}_j\right]_+=\mathbb{0}\end{cases}\Longrightarrow \boxed{\left[\underline{\alpha}_i,\underline{\alpha}_j\right]_+=2\delta_{ij}\mathbb{1}}$ Clifford algebra Coefficients of m^2: $\boxed{\underline{\beta}^2=\mathbb{1}}$ solved by $\boxed{\underline{\beta}=\gamma^0=\begin{pmatrix}\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & -\mathbb{1}_2\end{pmatrix}}$ Coefficients of $\mathbb{0}$: $\underline{\alpha}_i\underline{\beta}+\underline{\beta}\underline{\alpha}_i=\left[\underline{\alpha}_i,\underline{\beta}\right]_+=\mathbb{0}$ solved by $\boxed{\underline{\alpha}_i=\begin{pmatrix}\mathbb{0}_2 & \sigma_i\\ \sigma_i & \mathbb{0}_2\end{pmatrix}}$ with $\boxed{\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$
Final form	$i\partial_t\Psi=\hat{H}_D\Psi\overset{(3)}{\Rightarrow}i\partial_t\Psi=\left(\underline{\alpha}_i\hat{p}_i+\underline{\beta}m\right)\Psi\Longrightarrow i\partial_t\Psi=\left(\frac{1}{i}\underline{\alpha}_i\partial_i+\underline{\beta}m\right)\Psi\Longrightarrow i\partial_t\Psi-\frac{1}{i}\underline{\alpha}_i\partial_i\Psi-\underline{\beta}m\Psi=0\Longrightarrow$ $i\partial_t\Psi+i\underline{\alpha}_i\partial_i\Psi-\underline{\beta}m\Psi=0 \mid \underline{\beta}\cdot\Longrightarrow i\underline{\beta}\partial_t\Psi+i\underline{\beta}\underline{\alpha}_i\partial_i\Psi-\underline{\beta}^2m\Psi=0 \mid \boxed{\underline{\beta}\overset{\text{def}}{=}\gamma^0,\underline{\beta}\underline{\alpha}_i=\gamma^i,\underline{\beta}^2=\mathbb{1}}$ $i\gamma^0\partial_0\Psi+i\gamma^i\partial_i\Psi-\mathbb{1}m\Psi=0 \mid \gamma^\mu=(\gamma^0,\gamma^i)^T\Longrightarrow \boxed{(i\gamma^\mu\partial_\mu-\mathbb{1}m)\Psi=0}\Longrightarrow\boxed{(i\partial\!\!\!/-\mathbb{1}m)\Psi=0 \text{ with } \partial\!\!\!/\overset{\text{def}}{=}\gamma^\mu\partial_\mu,\ m \text{ ... rest mass}}$

2.2 Properties of γ^μ

Gamma Matrices (Dirac represent-tation)	$\gamma^0\overset{\text{def}}{=}\underline{\beta}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ $\gamma^1\overset{\text{def}}{=}\underline{\beta}\underline{\alpha}_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$ $\gamma^2\overset{\text{def}}{=}\underline{\beta}\underline{\alpha}_2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}$ Clifford algebra $\gamma^3\overset{\text{def}}{=}\underline{\beta}\underline{\alpha}_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$ $\gamma^5\overset{\text{def}}{=}i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ $(\gamma^0)^2=\underline{\beta}^2=\mathbb{1}$, $(\gamma^i)^2=-\mathbb{1}$, $\gamma^{0\dagger}=(\gamma^0)^{-1}=\gamma^0$, $(\gamma^i)^\dagger=-\gamma^i$ $[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}\mathbb{1}$, $(\gamma^5)^2=\mathbb{1}$, $\gamma^{5\dagger}=(\gamma^5)^{-1}=\gamma^5$, $\gamma^5\gamma^\mu=-\gamma^\mu\gamma^5$ $(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0$, $\gamma^0\gamma^1=-\gamma^1\gamma^0$, $\gamma^0\gamma^2=-\gamma^2\gamma^0$, $\gamma^0\gamma^3=-\gamma^3\gamma^0$
Proof $(\gamma^i)^2=-\mathbb{1}$	$(\gamma^i)^2=\underline{\beta}\underline{\alpha}_i\underline{\beta}\underline{\alpha}_i=\left(\underline{\beta}\underline{\alpha}_i+\underline{\alpha}_i\underline{\beta}-\underline{\alpha}_i\underline{\beta}\right)\underline{\beta}\underline{\alpha}_i=\left(\left[\underline{\beta},\underline{\alpha}_i\right]_+-\underline{\alpha}_i\underline{\beta}\right)\underline{\beta}\underline{\alpha}_k \mid \left[\underline{\beta},\underline{\alpha}_i\right]_+=\mathbb{0}\Longrightarrow$ $(\gamma^i)^2=-\underline{\alpha}_i\underline{\beta}\underline{\beta}\underline{\alpha}_i \mid \underline{\beta}\underline{\beta}=\underline{\beta}^2=\mathbb{1}\Longrightarrow(\gamma^i)^2=-\underline{\alpha}_i\underline{\alpha}_i \mid \underline{\alpha}_i\underline{\alpha}_i=\mathbb{1}\Longrightarrow\boxed{(\gamma^i)^2=-\mathbb{1}}$ Proof γ^0 hermitian: $(\gamma^0)^\dagger=\underline{\beta}^\dagger=\underline{\beta}\Longrightarrow(\gamma^0)^\dagger=\gamma^0$
Proof γ^i anti hermitian	$(\gamma^i)^\dagger=\left(\underline{\beta}\underline{\alpha}_i\right)^\dagger=\underline{\alpha}_i^\dagger\underline{\beta}^\dagger=\underline{\alpha}_i\underline{\beta}=\underline{\alpha}_i\underline{\beta}+\underline{\beta}\underline{\alpha}_i-\underline{\beta}\underline{\alpha}_i=\left[\underline{\alpha}_i,\underline{\beta}\right]_+-\underline{\beta}\underline{\alpha}_i \mid \left[\underline{\alpha}_i,\underline{\beta}\right]_+=\mathbb{0}\Longrightarrow\boxed{\underline{\alpha}_i\underline{\beta}=-\underline{\beta}\underline{\alpha}_i}$ $\boxed{(\gamma^i)^\dagger=-\underline{\beta}\underline{\alpha}_i=-\gamma^i}$
$\underline{\alpha}_i^2,\underline{\beta}^2$	$\underline{\alpha}_i^2=\underline{\beta}^2=\mathbb{1}\Longrightarrow$ Eigenvalues $=\pm1$ $\mid (\gamma^i)^2=-\mathbb{1}\Longrightarrow$ Eigenvalues $=\pm i$ $\mid \mathrm{tr}(\gamma^\mu\gamma^\nu)=4g^{\mu\nu}$
tracelessness	$\underline{\alpha}_i=\mathbb{1}\underline{\alpha}_i=\underline{\beta}\underline{\beta}\underline{\alpha}_i=-\underline{\beta}\underline{\alpha}_i\underline{\beta}\Longrightarrow\mathrm{tr}(\underline{\alpha}_i)=\mathrm{tr}\left(-\underline{\beta}\underline{\alpha}_i\underline{\beta}\right)=\mathrm{tr}\left(-\underline{\alpha}_i\underline{\beta}\underline{\beta}\right)=\mathrm{tr}(-\underline{\alpha}_i)\Longrightarrow\boxed{\mathrm{tr}(\underline{\alpha}_i)=0}$ $\boxed{\mathrm{tr}\left(\underline{\beta}\right)=0}$ $\boxed{\mathrm{tr}(\gamma^\mu)=0}$
Dirac repre-sentation:	$\underline{\alpha}_i=\begin{pmatrix}\mathbb{0}_2&\sigma_i\\ \sigma_i&\mathbb{0}_2\end{pmatrix},\underline{\beta}=\begin{pmatrix}\mathbb{1}_2&\mathbb{0}_2\\ \mathbb{0}_2&-\mathbb{1}_2\end{pmatrix},\gamma^i=\underline{\beta}\underline{\alpha}_i=\begin{pmatrix}\mathbb{0}_2&\sigma_i\\ -\sigma_i&\mathbb{0}_2\end{pmatrix}$ Weyl (chiral) represent.: $\gamma^0=\begin{pmatrix}\mathbb{0}_2&\mathbb{1}_2\\ \mathbb{1}_2&\mathbb{0}_2\end{pmatrix},\gamma^i=\begin{pmatrix}\mathbb{0}_2&\sigma_i\\ -\sigma_i&\mathbb{0}_2\end{pmatrix},\gamma^5=\begin{pmatrix}-\mathbb{1}_2&\mathbb{0}_2\\ \mathbb{0}_2&\mathbb{1}_2\end{pmatrix}$
Majorana represent.:	$\gamma^0=\begin{pmatrix}\mathbb{0}_2&\sigma_2\\ \sigma_2&\mathbb{0}_2\end{pmatrix},\gamma^1=\begin{pmatrix}i\sigma_3&\mathbb{0}_2\\ \mathbb{0}_2&i\sigma_3\end{pmatrix},\gamma^2=\begin{pmatrix}\mathbb{0}_2&-\sigma_2\\ \sigma_2&\mathbb{0}_2\end{pmatrix},\gamma^3=\begin{pmatrix}-i\sigma_1&\mathbb{0}_2\\ \mathbb{0}_2&-i\sigma_1\end{pmatrix}$ All γ-matrices are imaginary. Dirac equation becomes real. Easy descr. of neutral particles.

2.3 Dirac Adjoint Equation and Dirac Current

Adjoint spinor	Because Hermiticity properties of γ-matrices can be summarized $\boxed{(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0}$ it is natural to define the Dirac adjoint spinor by $\boxed{\overline{\Psi}\overset{\text{def}}{=}\Psi^\dagger\gamma^0}$ $(\gamma^0)^\dagger=\gamma^0\gamma^0\gamma^0=\mathbb{1}\gamma^0=\gamma^0$ $(\gamma^i)^\dagger=\gamma^0\gamma^i\gamma^0=-\gamma^i\gamma^0\gamma^0=-\gamma^i\mathbb{1}=-\gamma^i$
Adjoint Dirac equation	$(i\gamma^\mu\partial_\mu-m)\Psi=0\mid^\dagger\Longrightarrow\Psi^\dagger\left(-i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu-m\right)=0\mid(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0\Longrightarrow\Psi^\dagger\left(-i(\gamma^0\gamma^\mu\gamma^0)\overleftarrow{\partial}_\mu-m\gamma^0\gamma^0\right)=0\Longrightarrow$ $\Psi^\dagger\gamma^0\left(-i\gamma^\mu\overleftarrow{\partial}_\mu-m\right)\cancel{\gamma^0}=0\mid\Psi^\dagger\gamma^0\overset{\text{def}}{=}\overline{\Psi}\Longrightarrow\overline{\Psi}\left(-i\gamma^\mu\overleftarrow{\partial}_\mu-m\right)=0\mid\cdot(-1)\Longrightarrow\overline{\Psi}\left(i\gamma^\mu\overleftarrow{\partial}_\mu+m\right)=0\mid\gamma^\mu\overleftarrow{\partial}_\mu\overset{\text{def}}{=}\overleftarrow{\partial\!\!\!/}$ $\boxed{\overline{\Psi}\left(i\overleftarrow{\partial\!\!\!/}+m\right)=0 \text{ with } \overleftarrow{\partial\!\!\!/}\overset{\text{def}}{=}\gamma^\mu\overleftarrow{\partial}_\mu \text{ and } \overline{\Psi}\overset{\text{def}}{=}\Psi^\dagger\gamma^0}$
continuity equation	Dirac: $(i\partial\!\!\!/-m)\Psi=0\mid\overline{\Psi}\cdot\Longrightarrow\overline{\Psi}(i\partial\!\!\!/-m)\Psi=0$ ... (1), adjoint Dirac: $\overline{\Psi}(i\overleftarrow{\partial\!\!\!/}+m)=0\mid\cdot\Psi\Longrightarrow\overline{\Psi}\left(i\overleftarrow{\partial\!\!\!/}+m\right)\Psi=0$ ... (2) $(2)+(1)\Longrightarrow i\overline{\Psi}\overleftarrow{\partial\!\!\!/}\Psi+\cancel{\overline{\Psi}m\Psi}+i\overline{\Psi}\partial\!\!\!/\Psi-\cancel{\overline{\Psi}m\Psi}=0\mid:i\Longrightarrow\overline{\Psi}(\overleftarrow{\partial\!\!\!/}+\partial\!\!\!/)\Psi=0\Longrightarrow\overline{\Psi}(\gamma^\mu\overleftarrow{\partial}_\mu+\gamma^\mu\partial_\mu)\Psi=0\Longrightarrow$ $(\overline{\Psi}\gamma^\mu)\overleftarrow{\partial}_\mu\Psi+\overline{\Psi}\gamma^\mu\partial_\mu\Psi=0\mid(\overline{\Psi}\gamma^\mu)\overleftarrow{\partial}_\mu=\gamma^\mu\overline{\Psi}\overleftarrow{\partial}_\mu=\partial_\mu\overline{\Psi}\gamma^\mu\Longrightarrow\partial_\mu\overline{\Psi}\gamma^\mu\Psi+\overline{\Psi}\gamma^\mu\partial_\mu\Psi=0\Longrightarrow\boxed{\partial_\mu\left(\overline{\Psi}\gamma^\mu\Psi\right)=\partial_\mu j^\mu=0}$
Probability density	$\rho=j^0=\overline{\Psi}\gamma^0\Psi=\Psi^\dagger\gamma^0\gamma^0\Psi=\Psi^\dagger\mathbb{1}\Psi=\Psi^\dagger\Psi\Longrightarrow\boxed{\rho=\sum_{\alpha=1}^4\Psi_\alpha^\dagger\Psi_\alpha\geq0}$ ... positive definite

2.4 Covariance of Dirac Equation

Poincaré Trafo	$x'^\mu = L^\mu{}_\nu x^\nu + a^\mu$ | $g_{\mu\nu} = g_{\rho\sigma} L^\rho{}_\mu L^\sigma{}_\nu = L^\rho{}_\mu g_{\rho\sigma} L^\sigma{}_\nu = (L^T)_\mu{}^\rho g_{\rho\sigma} L^\sigma{}_\nu \Rightarrow \boxed{\underline{\underline{g}} = \underline{\underline{L}}^T \underline{\underline{g}}\,\underline{\underline{L}}}$
Invariance	$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\Psi(x^\sigma) = 0 \;\Leftrightarrow\; \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\Psi'(x'^\sigma) = 0$
Translations	$x^\mu \to x'^\mu = x^\mu + a^\mu \Rightarrow \frac{\partial x'^\mu}{\partial x^\mu} = 1 \Rightarrow \partial x'^\mu = \partial x^\mu \Rightarrow \boxed{\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\mu}} \Rightarrow$ Invariance trivially fulfilled by $\Psi'(x'^\mu) = \Psi(x^\mu) \Rightarrow \Psi'(x^\mu + a^\mu) = \Psi(x^\mu) = \Psi(x'^\mu - a^\mu) \Rightarrow \boxed{\Psi'(x^\mu) = \Psi(x^\mu - a^\mu)}$
Condition for covariant transformation of γ^μ	$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu \Rightarrow x'^\mu = L^\mu{}_\nu x^\nu \Rightarrow \frac{\partial x'^\mu}{\partial x^\nu} = L^\mu{}_\nu \Rightarrow \frac{\partial}{\partial x^\nu} = L^\mu{}_\nu \frac{\partial}{\partial x'^\mu} \Rightarrow \boxed{\frac{\partial}{\partial x'^\mu} = (L^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}}$ ... (1) Ansatz: $\boxed{\Psi'_\alpha(x'^\mu) = S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\mu)}$ ... (2) Dirac: $i\gamma^\mu \frac{\partial}{\partial x'^\mu}\Psi'(x'^\sigma) - m\,\Psi'(x'^\sigma) = 0 \overset{(2)}{\Rightarrow}$ $i\gamma^\mu \frac{\partial}{\partial x'^\mu}\left(S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\sigma)\right) - m\,S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\sigma) = 0 \overset{(1)}{\Rightarrow}$ $i\gamma^\mu (L^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}\left(S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\sigma)\right) - m\,S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\sigma) = 0 \Rightarrow$ $i\gamma^\mu (L^{-1})^\nu{}_\mu S_{\alpha\beta}(\underline{\underline{L}}) \frac{\partial}{\partial x^\nu}\Psi_\beta(x^\sigma) - m\,S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\sigma) = 0$ $i\gamma^\mu (L^{-1})^\nu{}_\mu \underline{\underline{S}}(\underline{\underline{L}}) \frac{\partial}{\partial x^\nu}\Psi(x^\sigma) - m\,\underline{\underline{S}}(\underline{\underline{L}})\,\Psi(x^\sigma) = 0 \,\Big\|\, \underline{\underline{S}}^{-1}(\underline{\underline{L}}) \cdot \Rightarrow$ $i\,\underbrace{\underline{\underline{S}}^{-1}(\underline{\underline{L}})\,\gamma^\mu (L^{-1})^\nu{}_\mu\,\underline{\underline{S}}(\underline{\underline{L}})}_{\gamma^\nu} \frac{\partial}{\partial x^\nu}\Psi(x^\sigma) - m\,\Psi(x^\sigma) = 0 \Rightarrow \underline{\underline{S}}^{-1}(\underline{\underline{L}})\,\gamma^\mu (L^{-1})^\nu{}_\mu\,\underline{\underline{S}}(\underline{\underline{L}}) = \gamma^\nu \,\Big\|\, L^\mu{}_\nu \cdot \Rightarrow$ $\boxed{\underline{\underline{S}}^{-1}(\underline{\underline{L}})\,\gamma^\mu\,\underline{\underline{S}}(\underline{\underline{L}}) = L^\mu{}_\nu \gamma^\nu}$ ... condition for $\underline{\underline{S}}(\underline{\underline{L}})$
Infinitesimal Lorentz Trafo	$L^\rho{}_\sigma = \delta^\rho_\sigma + \varepsilon\omega^\rho{}_\sigma$ with ε ... „small" ... (1) Invariance of Minkowski metric implies anti-symmetry of $\omega_{\mu\nu}$: $g_{\mu\nu} = L^\rho{}_\mu L^\sigma{}_\nu g_{\rho\sigma} \overset{(1)}{=} (\delta^\rho_\mu + \varepsilon\omega^\rho{}_\mu)(\delta^\sigma_\nu + \varepsilon\omega^\sigma{}_\nu) g_{\rho\sigma} = (\delta^\rho_\mu + \varepsilon\omega^\rho{}_\mu)(\delta^\sigma_\nu g_{\alpha\sigma} + \varepsilon\omega^\sigma{}_\nu g_{\alpha\sigma}) = (\delta^\rho_\mu + \varepsilon\omega^\rho{}_\mu)(g_{\rho\nu} + \varepsilon\omega_{\rho\nu})$ $g_{\mu\nu} = \delta^\rho_\mu g_{\rho\nu} + \rho_\mu g_{\rho\nu} + \delta^\rho_\mu \varepsilon\omega_{\rho\nu} + \varepsilon\omega^\rho{}_\mu \omega_{\rho\nu} \Rightarrow g_{\mu\nu} = g_{\mu\nu} + \varepsilon\omega_{\nu\mu} + \varepsilon\omega_{\mu\nu} + \mathcal{O}(\omega^2) \Rightarrow \omega_{\nu\mu} + \omega_{\mu\nu} = 0 \Rightarrow \boxed{\omega_{\mu\nu} = -\omega_{\nu\mu}}$
Generators active trafos.	In D spacetime dimensions there are $\frac{D(D-1)}{2}$ independent Lorentz trafos (e.g. 3 boosts and 3 rotations in (3+1) dimensions. $\omega^{inf}_{\mu\nu} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}_{\mu\nu} = \omega_{01}(L^{01})_{\mu\nu} + \omega_{02}(L^{02})_{\mu\nu} + \omega_{03}(L^{03})_{\mu\nu} + \omega_{12}(L^{12})_{\mu\nu} + \omega_{13}(L^{13})_{\mu\nu} + \omega_{23}(L^{23})_{\mu\nu}$ $(L^{01})_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{02})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{03})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{12})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{13})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{23})_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{\mu\nu}$ $(L^{\alpha\beta})_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$ $(L^{\alpha\beta})_{\mu\nu} = -(L^{\beta\alpha})_{\mu\nu} = -(L^{\alpha\beta})_{\nu\mu}$ Infinitesimal trafo from generators $L^\mu{}_\nu = \delta^\mu_\nu + \frac{1}{2}\omega_{\alpha\beta}(L^{\alpha\beta})^\mu{}_\nu \Leftrightarrow$ $\mathrm{L} = \mathbb{1} + \frac{1}{2}\omega_{\alpha\beta} L^{\alpha\beta}$
for passive: change signs	$(L^{01})^\mu{}_\nu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{02})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{03})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{12})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{13})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{23})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^\mu{}_\nu$ $(L^{\alpha\beta})^\mu{}_\nu = (L^{\alpha\beta})_{\gamma\nu} g^{\gamma\mu}$ (sign change in all „spatial lines" (line 1-3) Generators for active rotations and active boosts $(L^{01})^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{02})^{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{03})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{12})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{13})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{23})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\mu\nu}$ $(L^{\alpha\beta})^{\mu\nu} = (L^{\alpha\beta})_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu}$ (sign change in all „spatial lines" (line 1-3) and „spatial columns" (column 1-3)
Ansatz for $\underline{\underline{S}}$ (infinitesimal spinor trafo)	$\underline{\underline{S}} = \mathbb{1} + \varepsilon\underline{\underline{T}}$ ... (3) $\Rightarrow \underline{\underline{S}}^{-1} = \mathbb{1} - \varepsilon\underline{\underline{T}}$ because $\underline{\underline{S}}\,\underline{\underline{S}}^{-1} = (\mathbb{1} + \varepsilon\underline{\underline{T}})(\mathbb{1} - \varepsilon\underline{\underline{T}}) = \mathbb{1} + \varepsilon^2\underline{\underline{T}}^2 = \mathbb{1}$ $\overset{\text{condition for } \underline{\underline{S}}(\underline{\underline{L}})}{\Longrightarrow}$ $(\mathbb{1} - \varepsilon\underline{\underline{T}})\gamma^\mu(\mathbb{1} + \varepsilon\underline{\underline{T}}) = (\delta^\mu_\nu + \varepsilon\omega^\mu{}_\nu)\gamma^\nu \Rightarrow (\mathbb{1} - \varepsilon\underline{\underline{T}})(\gamma^\mu + \varepsilon\gamma^\mu\underline{\underline{T}}) = \delta^\mu_\nu\gamma^\nu + \varepsilon\omega^\mu{}_\nu\gamma^\nu \Rightarrow$ $\cancel{\gamma^\mu} + \varepsilon\gamma^\mu\underline{\underline{T}} - \varepsilon\underline{\underline{T}}\gamma^\mu - \cancel{\varepsilon^2\underline{\underline{T}}\gamma^\mu\underline{\underline{T}}} = \cancel{\gamma^\mu} + \varepsilon\omega^\mu{}_\nu\gamma^\nu \Rightarrow \varepsilon(\gamma^\mu\underline{\underline{T}} - \underline{\underline{T}}\gamma^\mu) = \varepsilon\omega^\mu{}_\nu\gamma^\nu \Rightarrow \boxed{[\gamma^\mu, \underline{\underline{T}}] = \omega^\mu{}_\nu\gamma^\nu}$ ... (4)
infinitesimal spinor transformation	(4) is solved by $\underline{\underline{T}} = -\frac{i}{2}\omega^{inf}_{\mu\nu} S^{\mu\nu}$ with $\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$ ...generators $\Rightarrow$ $\underline{\underline{T}} = \frac{1}{8}\omega^{inf}_{\mu\nu}[\gamma^\mu, \gamma^\nu] \overset{(3)}{\Rightarrow}$ $\boxed{\underline{\underline{S}}_{inf} = \mathbb{1} + \frac{\varepsilon}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]}$ to be used in $\boxed{\Psi'_\alpha(x'^\mu) = S_{\alpha\beta}(\underline{\underline{L}})\,\Psi_\beta(x^\mu)}$
finite LT	$L^\mu{}_\nu = \lim_{n\to\infty}\left(\delta^\mu_\nu + \frac{\xi}{n}\omega^\mu{}_\nu\right)^n \Rightarrow \boxed{L^\mu{}_\nu = e^{\xi\omega^\mu{}_\nu}}$
finite spinor trafo	$\underline{\underline{S}} = e^{\frac{1}{8}\xi\omega^{inf}_{\mu\nu}[\gamma^\mu, \gamma^\nu]} \,\Big\|\, \omega_{\mu\nu} \overset{\text{def}}{=} \xi\omega^{inf}_{\mu\nu} \Rightarrow \boxed{\underline{\underline{S}} = e^{\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]}} = e^{-\frac{i}{2}\omega_{\mu\nu}\frac{i}{4}[\gamma^\mu, \gamma^\nu]} \,\Big\|\, S^{\mu\nu} \overset{\text{def}}{=} \frac{i}{4}[\gamma^\mu, \gamma^\nu] \Rightarrow \boxed{\underline{\underline{S}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}}$ in dirac repr. $\boxed{\underline{\underline{S}}^{-1} = \gamma^0\underline{\underline{S}}^\dagger\gamma^0}$

2.4.1 Rotation

Passive rotation around z-axis	$\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=\underline{\underline{R}}_z\begin{pmatrix}x\\y\\z\end{pmatrix}$ with $\underline{\underline{R}}_z=\begin{pmatrix}\cos(\varphi)&\sin(\varphi)&0\\-\sin(\varphi)&\cos(\varphi)&0\\0&0&1\end{pmatrix}\mid \begin{matrix}\cos(\varepsilon)\approx 1\\ \sin(\varepsilon)\approx\varepsilon\end{matrix}\Rightarrow \underline{\underline{R}}_z^{inf}=\begin{pmatrix}1&\varepsilon&0\\-\varepsilon&1&0\\0&0&1\end{pmatrix}$ so that $\begin{matrix}x'=x+\varepsilon y\\y'=y-\varepsilon x\\z'=z\end{matrix}$
Lorentz trafo	$L^{\mu}{}_{\nu}=\begin{pmatrix}1&0&0&0\\0&\cos(\varphi)&\sin(\varphi)&0\\0&-\sin(\varphi)&\cos(\varphi)&0\\0&0&0&1\end{pmatrix}\mid\begin{matrix}\cos(\varepsilon)\approx 1\\ \sin(\varepsilon)\approx\varepsilon\end{matrix}\Rightarrow L_{inf}{}^{\mu}{}_{\nu}=\begin{pmatrix}1&0&0&0\\0&1&\varepsilon&0\\0&-\varepsilon&1&0\\0&0&0&1\end{pmatrix}=\mathbb{1}+\varepsilon\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}=\mathbb{1}+\varepsilon\omega^{\mu}{}_{\nu}$
infinitesimal spinor trafo	$\underline{\underline{S}}_{\text{inf}}=\mathbb{1}+\frac{\varepsilon}{8}\omega_{\mu\nu}[\gamma^{\mu},\gamma^{\nu}]=\mathbb{1}-\frac{\varepsilon}{4}[\gamma^1,\gamma^2]\Rightarrow \underline{\underline{S}}_{\text{inf}}=\mathbb{1}+\varepsilon\frac{i}{2}\Sigma_3$ with $\Sigma_3=\sigma_3\oplus\sigma_3=\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}$ ... (1)
Deriving Spin ½	$\Psi'_{\alpha}(x'^{\mu})=S_{\alpha\beta}\Psi_{\beta}(x^{\mu})\Rightarrow\Psi'_{\alpha}(\vec{x}',t')=S_{\alpha\beta}\Psi_{\beta}(\vec{x},t)\mid t'=t\Rightarrow\Psi'_{\alpha}(\vec{x}',t)=S_{\alpha\beta}\Psi_{\beta}(\vec{x},t)\Rightarrow$ $\Psi'_{\alpha}(\underline{\underline{R}}\vec{x},t)=S_{\alpha\beta}\Psi_{\beta}(\underline{\underline{R}}^{-1}\underline{\underline{R}}\vec{x},t)\mid\underline{\underline{R}}\vec{x}\stackrel{\text{def}}{=}\tilde{\vec{x}}\Rightarrow\Psi'_{\alpha}(\tilde{\vec{x}},t)=S_{\alpha\beta}\Psi_{\beta}(\underline{\underline{R}}^{-1}\tilde{\vec{x}},t)\mid\tilde{\vec{x}}\to\vec{x}\Rightarrow\Psi'_{\alpha}(\vec{x},t)=S_{\alpha\beta}\Psi_{\beta}(\underline{\underline{R}}^{-1}\vec{x},t)\stackrel{(1)}{\Rightarrow}$ $\Psi'(\vec{x},t)=\left(\mathbb{1}+\varepsilon\frac{i}{2}\Sigma_3\right)\Psi(\underline{\underline{R}}_{\text{inf}}^{-1}\vec{x},t)\mid\underline{\underline{R}}_{\text{inf}}^{-1}\vec{x}=(x-\varepsilon y,y-\varepsilon y,z)^T\Rightarrow\Psi'_{\alpha}(\vec{x},t)=\left(\mathbb{1}+\varepsilon\frac{i}{2}\Sigma_3\right)\Psi(x-\varepsilon y,y+\varepsilon x,z,t)\Rightarrow$ $\Psi'(\vec{x},t)=\left(\mathbb{1}+\varepsilon\frac{i}{2}\Sigma_3\right)\left(\Psi(\vec{x},t)-\varepsilon y\frac{\partial}{\partial x}\Psi(\vec{x},t)+\varepsilon x\frac{\partial}{\partial y}\Psi(\vec{x},t)\right)$ $\Psi'(\vec{x},t)=\Psi(\vec{x},t)-\varepsilon y\frac{\partial}{\partial x}\Psi(\vec{x},t)+\varepsilon x\frac{\partial}{\partial y}\Psi(\vec{x},t)+\varepsilon\frac{i}{2}\Sigma_3\Psi(\vec{x},t)\cancel{-\varepsilon^2\frac{i}{2}\Sigma_3 y\frac{\partial}{\partial x}\Psi(\vec{x},t)+\varepsilon^2\frac{i}{2}\Sigma_3 x\frac{\partial}{\partial y}\Psi(\vec{x},t)}$ $\Psi'(\vec{x},t)=\Psi(\vec{x},t)+i\varepsilon\Bigg(\underbrace{\underbrace{\frac{1}{2}\Sigma_3}_{spin\ s_z}+\underbrace{x\frac{\partial}{i\partial y}-y\frac{\partial}{i\partial x}}_{l_z}}_{J_z}\Bigg)\Psi(\vec{x},t)$ Because Σ_3 has eigenvalues ± 1, this shows that Ψ carries spin ½
Finite rotation Spinor Trafo	$\underline{\underline{S}}=\lim_{n\to\infty}\left(\mathbb{1}+\frac{\varphi}{n}\frac{i}{2}\Sigma_3\right)^n\Rightarrow\underline{\underline{S}}=e^{i\frac{\varphi}{2}\Sigma_3}\Rightarrow\Psi'(\vec{x}',t')=e^{i\frac{\varphi}{2}\Sigma_3}\Psi(\vec{x},t)=\left(\mathbb{1}\cos\left(\frac{\varphi}{2}\right)+i\Sigma_3\sin\left(\frac{\varphi}{2}\right)\right)\Psi(\vec{x},t)$ $\underline{\underline{S}}(\varphi=2\pi)=-\mathbb{1}\Rightarrow$ only a 4π rotation brings us back to $\underline{\underline{S}}(\varphi=4\pi)=\mathbb{1}$

2.4.2 Lorentz Boost

Boost in x-direction infinitesimal Lorentz Trafo	$L_{inf}{}^{\mu}{}_{\nu}=\begin{pmatrix}1&\varepsilon&0&0\\\varepsilon&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\mathbb{1}+\varepsilon\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\mathbb{1}+\varepsilon\omega^{\mu}{}_{\nu}$ with $\omega^{\mu}{}_{\nu}=(L^{01})^{\mu}{}_{\nu}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ $(\omega^{\mu}{}_{\nu})^2=\mathbb{1}_2\oplus\mathbb{0}_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad(\omega^{\mu}{}_{\nu})^3=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\omega^{\mu}{}_{\nu},\ldots$
Finite Lorentz Trafo	$L^{\mu}{}_{\nu}=e^{\xi\omega^{\mu}{}_{\nu}}=\sum_{n=0}^{\infty}\omega^n\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\omega^{2n}\frac{\xi^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\omega^{2n+1}\frac{\xi^{2n+1}}{(2n+1)!}=\omega^2\sum_{n=0}^{\infty}\frac{\xi^{2n}}{(2n)!}+\omega\sum_{n=0}^{\infty}\frac{\xi^{2n+1}}{(2n+1)!}$ $L^{\mu}{}_{\nu}=1-\omega^2+\omega^2\cosh(\xi)+\omega\sinh(\xi)=\begin{pmatrix}\cosh(\xi)&-\sinh(\xi)&0&0\\-\sinh(\xi)&\cosh(\xi)&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}\gamma&-\beta\gamma&0&0\\-\beta\gamma&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ ξ ... rapidity
Finite Boost Spinor Trafo	$\underline{\underline{S}}=e^{\frac{1}{8}\xi\omega_{\mu\nu}^{inf}[\gamma^{\mu},\gamma^{\nu}]}=e^{-\frac{1}{4}\xi[\gamma^0,\gamma^1]}$ with $[\gamma^0,\gamma^1]=2\alpha_1=2\begin{pmatrix}0&\sigma_1\\\sigma_1&0\end{pmatrix}$ $\underline{\underline{S}}=\mathbb{1}\cosh\left(\frac{\xi}{2}\right)-\alpha_1\sinh\left(\frac{\xi}{2}\right)=\mathbb{1}\cosh\left(\frac{\xi}{2}\right)-\begin{pmatrix}0&\sigma_1\\\sigma_1&0\end{pmatrix}\sinh\left(\frac{\xi}{2}\right)$ $\underline{\underline{S}}=\begin{pmatrix}\cosh\left(\frac{\xi}{2}\right)&0&0&-\sinh\left(\frac{\xi}{2}\right)\\0&\cosh\left(\frac{\xi}{2}\right)&-\sinh\left(\frac{\xi}{2}\right)&0\\0&-\sinh\left(\frac{\xi}{2}\right)&\cosh\left(\frac{\xi}{2}\right)&0\\-\sinh\left(\frac{\xi}{2}\right)&0&0&\cosh\left(\frac{\xi}{2}\right)\end{pmatrix}$ $\tanh\left(\frac{\xi}{2}\right)=\frac{\tanh(\xi)}{1+\sqrt{1-\tanh^2(\xi)}}=\frac{v}{1+\sqrt{1-v^2}}=\frac{p}{E+m};\ \cosh\left(\frac{\xi}{2}\right)=\sqrt{\frac{E+m}{2m}};\ \sinh\left(\frac{\xi}{2}\right)=\tanh\left(\frac{\xi}{2}\right)\cosh\left(\frac{\xi}{2}\right)$

2.4.3 Components of the Lorentz Group

Reducibility	The Lorentz Trafo $L^\mu{}_\nu = e^{\xi\omega^\mu{}_\nu}$ is continuously connected with the identity. This does not exhaust all possibilities of the Lorentz Group, but only those with $\det(\underline{L}) = +1$ and $L^0{}_0 \geq 1$ This subgroup of Lorentz trafos is called *proper orthochronous Lorentz group* $\mathcal{L}_+^\uparrow$ As long as only elements of $\mathcal{L}_+^\uparrow$ are considered, $\underline{S}(\underline{L})$ is reducible, i.e. ∃ nontrivial, invariant subspaces Visible e.g in chiral representation $\gamma^0 = \begin{pmatrix} \mathbb{0}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{0}_2\end{pmatrix}, \gamma^i = \begin{pmatrix}\mathbb{0}_2 & \sigma_i \\ -\sigma_i & \mathbb{0}_2\end{pmatrix}$
Consequences in chiral Weyl represent.:	In the chiral representation of the Dirac Matrices the generators $S^{\mu\nu}$of $\underline{S}(\underline{L}) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$ (with $\xi\omega_{\mu\nu} \to \omega_{\mu\nu}$) read $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] \Longrightarrow \boxed{S^{\mu\mu} = \mathbb{0}, S^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = -\frac{i}{2}\begin{pmatrix}\sigma_i & \mathbb{0}_2\\ \mathbb{0}_2 & -\sigma_i\end{pmatrix}; S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix}\sigma_k & \mathbb{0}_2 \\ \mathbb{0}_2 & \sigma_k\end{pmatrix}}$ ... block diagonal when applying $\Psi' = \underline{S}(\underline{L})\,\Psi$, then the upper two components of Ψ never mix with the lower two components
Weyl equation (massless particle)	We start with Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ ... (1) using chiral representation $\gamma^0 = \begin{pmatrix} \mathbb{0}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{0}_2\end{pmatrix}, \gamma^i = \begin{pmatrix}\mathbb{0}_2 & \sigma_i \\ -\sigma_i & \mathbb{0}_2\end{pmatrix}$ Because of $p_\mu = \begin{pmatrix}\frac{E}{c}\\ -\vec{p}\end{pmatrix}$ ... (2) and $\hat{p}_\mu = \begin{pmatrix}\frac{i\hbar}{c}\partial_t \\ i\hbar\vec{\nabla}\end{pmatrix} \Big\vert \begin{matrix} c \overset{\text{def}}{=} 1 \\ \hbar \overset{\text{def}}{=} 1\end{matrix} \Longrightarrow \boxed{\hat{p}_\mu = \begin{pmatrix} i\partial_t \\ i\vec{\nabla}\end{pmatrix} = i\partial_\mu}$ ... (3) compare (2), (3) $\Longrightarrow \vec{p} \mathrel{\hat{=}} \hat{p}^i = -i\vec{\nabla} = -\hat{p}_i$ ... (4) $\quad m = 0 \overset{(1)}{\Rightarrow} i\gamma^\mu\partial_\mu\Psi = 0 \overset{(3)}{\Rightarrow} \boxed{\gamma^\mu\hat{p}_\mu\Psi = \hat{\not{p}}\Psi = 0} \Longrightarrow (\gamma^0\hat{p}_0 + \gamma^i\hat{p}_i)\Psi = 0 \overset{(4)}{\Rightarrow}$ $(\gamma^0\hat{p}_0 - \gamma^i\hat{p}^i)\Psi = 0 \Big\vert \Psi \overset{\text{def}}{=} \begin{pmatrix}\phi\\ \chi\end{pmatrix} \Longrightarrow (\gamma^0\hat{p}_0 - \gamma^i\hat{p}^i)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \Longrightarrow \begin{pmatrix}\mathbb{0}_2 & \mathbb{1}_2\\ \mathbb{1}_2 & \mathbb{0}_2\end{pmatrix}\hat{p}_0 - \begin{pmatrix}\mathbb{0}_2 & \sigma_i\\ -\sigma_i & \mathbb{0}_2\end{pmatrix}\hat{p}^i\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \Longrightarrow$ $(\mathbb{1}_2\hat{p}_0 + \hat{p}^i\sigma_i)\phi = 0, (\mathbb{1}_2\hat{p}_0 - \hat{p}^i\sigma_i)\chi = 0 \Longrightarrow \boxed{(\hat{p}_0 + \vec{p}\cdot\vec{\sigma})\phi = 0, (\hat{p}_0 - \vec{p}\cdot\vec{\sigma})\chi = 0}$
Full L. group	obtained by composing elements of $\mathcal{L}_+^\uparrow$ with $\hat{P}$ ... space inversion (parity Trafo), $\hat{T}$ ... time reversal, or $\hat{P}\hat{T}$ ... both
Parity Trafo	$\boxed{\hat{L}^\mu{}_\nu = \begin{pmatrix}1 & \\ & -\mathbb{1}_3\end{pmatrix}}$ condition for $\underline{S}(\underline{L}) = \underline{P}$: $\underline{S}^{-1}(\underline{L})\,\gamma^\mu\,\underline{S}(\underline{L}) = L^\mu{}_\nu\gamma^\nu \Longrightarrow \underline{P}^{-1}\gamma^\mu\underline{P} = \begin{pmatrix}1 & \\ & -\mathbb{1}_3\end{pmatrix}^\mu{}_\nu \gamma^\nu \Longrightarrow$ $\boxed{\underline{P}^{-1}\gamma^0\underline{P} = \gamma^0;\ \underline{P}^{-1}\gamma^i\underline{P} = -\gamma^i} \Longrightarrow \boxed{\underline{S}(\underline{L}) = \underline{P} = \pm\gamma^0}$ because $\gamma^0\underbrace{\gamma^0\gamma^0}_{\mathbb{1}} = \gamma^0;\ \underbrace{\gamma^0\gamma^i}_{-\gamma^i\gamma^0}\gamma^0 = -\gamma^i\underbrace{\gamma^0\gamma^0}_{\mathbb{1}} = -\gamma^i$
Parity violat.	chiral representation $\underline{P} = \gamma^0 = \begin{pmatrix}\mathbb{0}_2 & \mathbb{1}_2\\ \mathbb{1}_2 & \mathbb{0}_2\end{pmatrix} \Longrightarrow \underline{P}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}\chi\\ \phi\end{pmatrix} \Longrightarrow \chi, \phi$ get mixed, parity violation with Weyl spinor
$\hat{P}\hat{T}$ Trafo	$\boxed{\hat{L}^\mu{}_\nu = -\delta^\mu{}_\nu}$ condition for $\underline{S}(\underline{PT}) = \underline{P}$: $\underline{S}^{-1}(\hat{P}\hat{T})\,\gamma^\mu\,\underline{S}(\hat{P}\hat{T}) = -\delta^\mu{}_\nu\gamma^\nu = -\gamma^\mu \Longrightarrow$ $\boxed{\underline{S}(\hat{P}\hat{T}) = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma}$ Proof: $(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 = i\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^2\gamma^3\gamma^1\gamma^0$ $= i\gamma^2\gamma^1\gamma^3\gamma^0 = -i\gamma^2\gamma^1\gamma^0\gamma^3 = i\gamma^1\gamma^2\gamma^0\gamma^3 = -i\gamma^1\gamma^0\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$
Properties of γ^5	$\boxed{(\gamma^5)^2 = \mathbb{1}}\ \boxed{(\gamma^5)^\dagger = \gamma^5}\ \boxed{\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \Leftrightarrow [\gamma^5,\gamma^\mu]_+ = 0}\ \boxed{[\gamma^5, S^{\mu\nu}] = \left[\gamma^5, \frac{i}{4}[\gamma^\mu,\gamma^\nu]\right] = 0}$ Proof: $(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 = i\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^2\gamma^3\gamma^1\gamma^0$ $= i\gamma^2\gamma^1\gamma^3\gamma^0 = -i\gamma^2\gamma^1\gamma^0\gamma^3 = i\gamma^1\gamma^2\gamma^0\gamma^3 = -i\gamma^1\gamma^0\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$
Schurs lemma	If a matrix commutes with all elements of a matrix representation of a group, then it is either proportional to the unit matrix, or this representation is reducible. $\Longrightarrow S^{\mu\nu}$ is reducible $\Longrightarrow \underline{S}(\underline{L} \in \mathcal{L}_+^\uparrow) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$ is reducible
Chirality	The invariant subspaces are given by the projectors $\boxed{P_L = \frac{1}{2}(\mathbb{1} - \gamma^5),\ P_R = \frac{1}{2}(\mathbb{1} + \gamma^5)}$ ... left- and right-handed chirality $P_L^2 = \frac{1}{4}(\mathbb{1} - \gamma^5)(\mathbb{1} - \gamma^5) = \frac{1}{4}(\mathbb{1} - 2\gamma^5 + (\gamma^5)^2) \Big\vert (\gamma^5)^2 = \mathbb{1} \Longrightarrow P_L^2 = \frac{1}{4}(2\cdot\mathbb{1} - 2\gamma^5) = \frac{1}{2}(\mathbb{1} - \gamma^5) \Longrightarrow P_L^2 = P_L$ ... projector $\Psi_R = P_R\Psi$ is eigenstate of γ^5 with eigenvalue $\lambda = -1 \Longrightarrow \boxed{\gamma^5\Psi_L = -\Psi_L}$ Proof: $\gamma^5\Psi_L = \gamma^5 P_L\Psi = \gamma^5\frac{1}{2}(\mathbb{1} - \gamma^5)\Psi = \frac{1}{2}(\gamma^5 - \gamma^5\gamma^5)\Psi = \frac{1}{2}(\gamma^5 - \mathbb{1})\Psi = -\frac{1}{2}(\mathbb{1} - \gamma^5) = -P_L\Psi = -\Psi_L$ $\Psi_L = P_L\Psi$ is eigenstate of γ^5 with eigenvalue $\lambda = +1 \Longrightarrow \boxed{\gamma^5\Psi_R = +\Psi_R}$ Proof: $\gamma^5\Psi_R = \gamma^5 P_R\Psi = \gamma^5\frac{1}{2}(\mathbb{1} + \gamma^5)\Psi = \frac{1}{2}(\gamma^5 + \gamma^5\gamma^5)\Psi = \frac{1}{2}(\gamma^5 + \mathbb{1})\Psi = \frac{1}{2}(\mathbb{1} + \gamma^5) = P_R\Psi = +\Psi_R$ Weyl eq.: $\hat{\not{p}}\Psi = 0 \Longrightarrow \gamma^\mu\hat{p}_\mu\Psi = 0 \big\vert \gamma^5\gamma^0\cdot \Longrightarrow \gamma^5\gamma^0\gamma^\mu\hat{p}_\mu\Psi = 0 \Longrightarrow \gamma^5\gamma^0\gamma^\mu\begin{pmatrix}p^0\\ -\vec{p}\end{pmatrix}\Psi = 0$ $\boxed{\gamma^5 p^0\Psi = \vec{\Sigma}\cdot\vec{\mathrm{p}}\Psi}$ ... (5) with $\boxed{\vec{\Sigma} = (\Sigma_1,\Sigma_2,\Sigma_3)^T \text{ and } s_i = \frac{1}{2}\Sigma_i \text{ and } s_i = \frac{i}{8}\varepsilon_{ijk}[\gamma^j,\gamma^k]}$ $p^0 = E = \lvert\vec{p}\rvert \overset{(5)}{\Rightarrow} \gamma^5\lvert\vec{p}\rvert\Psi = \vec{\Sigma}\cdot\vec{\mathrm{p}}\Psi \Longrightarrow \gamma^5\Psi = \frac{\vec{\Sigma}\cdot\vec{\mathrm{p}}}{\lvert\vec{p}\rvert}\Psi \Longrightarrow \frac{1}{2}\gamma^5\Psi = \frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{\mathrm{p}}}{\lvert\vec{p}\rvert}\Psi \Longrightarrow$ $\boxed{\frac{1}{2}\gamma^5\Psi = \frac{\vec{s}\cdot\vec{\mathrm{p}}}{\lvert\vec{p}\rvert}\Psi}$ with $\frac{1}{2}\gamma^5$ ... chirality, $\frac{\vec{s}\cdot\vec{\mathrm{p}}}{\lvert\vec{p}\rvert}$ ... helicity of massless particle (boost invariant, changes under parity) Chirality is ½ times the eigenvalue of γ^5 and equals the helicity (projection of spin on the direction of $\vec{\mathrm{p}}$) of massless particles. In chiral representation: $\gamma^5 = \begin{pmatrix}-\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{1}_2\end{pmatrix} \Longrightarrow P_L = \frac{1}{2}(\mathbb{1} - \gamma^5) = \begin{pmatrix}\mathbb{1}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{0}_2\end{pmatrix}, P_R = \frac{1}{2}(\mathbb{1} + \gamma^5) = \begin{pmatrix}\mathbb{0}_2 & \mathbb{0}_2\\ \mathbb{0}_2 & \mathbb{1}_2\end{pmatrix}$

2.4.4 Dirac Field Bilinears and Higher Spin

Lorentz scalar $\overline{\Psi}\Psi$	$\Psi \to \Psi' = \underline{\underline{S}}\Psi \vert^{\dagger} \Longrightarrow \Psi^{\dagger} \to (\Psi')^{\dagger} = (\underline{\underline{S}}\Psi)^{\dagger} = \Psi^{\dagger}\underline{\underline{S}}^{\dagger} \vert \cdot \gamma^0 \Longrightarrow \Psi^{\dagger}\gamma^0 \to (\Psi')^{\dagger}\gamma^0 = \Psi^{\dagger}\underline{\underline{S}}^{\dagger}\gamma^0 = \Psi^{\dagger}\mathbb{1}\underline{\underline{S}}^{\dagger}\gamma^0 \vert \mathbb{1} = \gamma^0\gamma^0 \Longrightarrow$ $\Psi^{\dagger}\gamma^0 \to (\Psi')^{\dagger}\gamma^0 = \Psi^{\dagger}\gamma^0\gamma^0\underline{\underline{S}}^{\dagger}\gamma^0 \vert \Psi^{\dagger}\gamma^0 \stackrel{\text{def}}{=} \overline{\Psi} \Longrightarrow \boxed{\overline{\Psi} \to \overline{\Psi}' = \overline{\Psi}\gamma^0\underline{\underline{S}}^{\dagger}\gamma^0 = \overline{\Psi}\underline{\underline{S}}^{-1}} \overline{\Psi}\Psi \to \overline{\Psi}'\Psi' = \overline{\Psi}\underline{\underline{S}}^{-1}\underline{\underline{S}}\Psi \Longrightarrow \boxed{\overline{\Psi}\Psi \to \overline{\Psi}\Psi}$
prob. current vector $\overline{\Psi}\gamma^{\mu}\Psi$	$\boxed{j^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi} \to j'^{\mu} = \overline{\Psi}\underline{\underline{S}}^{-1}\gamma^{\mu}\underline{\underline{S}}\Psi \vert \underline{\underline{S}}^{-1}\gamma^{\mu}\underline{\underline{S}} = L^{\mu}{}_{\nu}\gamma^{\nu} \Longrightarrow j^{\mu} \to j'^{\mu} = \overline{\Psi}L^{\mu}{}_{\nu}\gamma^{\nu}\Psi = L^{\mu}{}_{\nu}\overline{\Psi}\gamma^{\nu}\Psi = L^{\mu}{}_{\nu}j^{\mu} \Longrightarrow \boxed{j^{\mu} \to j'^{\mu} = L^{\mu}{}_{\nu}j^{\mu}}$ Generally: $\boxed{\overline{\Psi}\gamma^{\mu_1} \dots \gamma^{\mu_k}\Psi}$ transforms as a tensor of rank k. Inserting γ^5 does not change that under $L \in \mathcal{L}_{+}^{\uparrow}$ $\boxed{\overline{\Psi}\gamma^{\mu}\gamma^5\Psi}$ ... axial pseudo-vector
Pseudo scalar $\overline{\Psi}\gamma^5\Psi$ under parity trafo	$\overline{\Psi}\gamma^5\Psi$ under parity transformation: $\overline{\Psi}'(x'^{\mu})\gamma^5\Psi'(x'^{\mu}) = \overline{\Psi}(x^{\mu})\hat{P}^{-1}\gamma^5\hat{P}\Psi(x^{\mu}) \vert \hat{P} = \gamma^0 \Longrightarrow$ $\overline{\Psi}'(x'^{\mu})\gamma^5\Psi'(x'^{\mu}) = \overline{\Psi}(x^{\mu})(\gamma^0)^{-1}\gamma^5\gamma^0\Psi(x^{\mu}) \vert (\gamma^0)^{-1} = \gamma^0 \Longrightarrow \overline{\Psi}'(x'^{\mu})\gamma^5\Psi'(x'^{\mu}) = \overline{\Psi}(x^{\mu})\gamma^0\gamma^5\gamma^0\Psi(x^{\mu}) \vert \gamma^5\gamma^0 = -\gamma^0\gamma^5 \Longrightarrow$ $\overline{\Psi}'(x'^{\mu})\gamma^5\Psi'(x'^{\mu}) = -\overline{\Psi}(x^{\mu})\gamma^0\gamma^0\gamma^5\Psi(x^{\mu}) \vert \gamma^0\gamma^0 = \mathbb{1} \Longrightarrow \boxed{\overline{\Psi}'(x'^{\mu})\gamma^5\Psi'(x'^{\mu}) = -\overline{\Psi}(x^{\mu})\gamma^5\Psi(x^{\mu})}$ ... pseudo-scalar
Decomposition of spinor matrices	Any 4x4 spinor matrix Γ can be decomposed in a linear combination of the following $16 = 4^2$ matrices and thus into terms with well-defined Lorentz transformation properties when sandwiched by Dirac fields: $\Gamma_S = \mathbb{1}$ (1 scalar), $\Gamma_V^{\mu} = \gamma^{\mu}$ (4 vector), $\Gamma_T^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}] \stackrel{\text{def}}{=} \sigma^{\mu\nu}$ (6 asym. tensor), $\Gamma_A^{\mu} = \gamma^{\mu}\gamma^5$ (4 axial vector), $\Gamma_P = \gamma^5$ (1 pseudoscalar). In N dimensions 2^N matrices
Rarita Schwinger eq.	Vector spinor with Lorentz-index Ψ^{μ} contains spin $\frac{1}{2}$ and $\frac{3}{2}$. $\tilde{\Psi} = \gamma^{\mu}\Psi_{\mu}$ projects out the spin ½ Dirac spinor. Rarita Schwinger equations: $\boxed{(i\partial\!\!\!/ - m)\Psi_{\mu} = 0 \text{ with } \partial\!\!\!/ \stackrel{\text{def}}{=} \gamma^{\mu}\partial_{\mu} \text{ and } \gamma^{\mu}\Psi_{\mu} = 0}$

3 Solutions of the Klein Gordon and Dirac Equation (plain wave, at rest)

Klein Gordon Equation: Plane wave solution	$(\Box + m^2)\Psi = 0 \Longrightarrow (\partial_{\nu}\partial^{\nu} + m^2)\Psi = 0 \vert \Psi \propto e^{ik_{\mu}x^{\mu}} \Longrightarrow (\partial_{\nu}\partial^{\nu} + m^2)e^{ik_{\mu}x^{\mu}} = 0 \Longrightarrow \partial_{\nu}\partial^{\nu}e^{ik_{\mu}x^{\mu}} + m^2e^{ik_{\mu}x^{\mu}} = 0 \Longrightarrow$ $\partial_{\nu}(ik_{\mu}\partial^{\nu}x^{\mu}e^{ik_{\mu}x^{\mu}}) + m^2e^{ik_{\mu}x^{\mu}} = 0 \vert \partial^{\nu}x^{\mu} = g^{\nu\mu} \Longrightarrow \partial_{\nu}(ik_{\mu}g^{\nu\mu}e^{ik_{\mu}x^{\mu}}) + m^2e^{ik_{\mu}x^{\mu}} = 0 \Longrightarrow$ $\partial_{\nu}(ik^{\nu}e^{ik_{\mu}x^{\mu}}) + m^2e^{ik_{\mu}x^{\mu}} = 0 \Longrightarrow ik^{\nu}(ik_{\mu}\partial_{\nu}x^{\mu})e^{ik_{\mu}x^{\mu}} + m^2e^{ik_{\mu}x^{\mu}} = 0 \vert \Longrightarrow \partial_{\nu}x^{\mu} = \delta_{\nu}^{\mu} = g^{\mu}{}_{\nu} \Longrightarrow$ $-k^{\nu}(k_{\mu}g^{\mu}{}_{\nu})e^{ik_{\mu}x^{\mu}} + m^2e^{ik_{\mu}x^{\mu}} = 0 \Longrightarrow -k^{\nu}k_{\nu}e^{ik_{\mu}x^{\mu}} + m^2e^{ik_{\mu}x^{\mu}} = 0 \vert k^{\nu}k_{\nu} = p^{\nu}p_{\nu} = m^2 \Longrightarrow$ $(-m^2 + m^2)e^{\mp ik_{\mu}x^{\mu}} = 0$ ■
Pos. and neg. Energy	Schrödinger: $i\partial_t\Psi = \hat{H}\Psi \vert \Psi \propto e^{ik_{\mu}x^{\mu}} \stackrel{\text{def}}{=} e^{\mp(ik_0x^0 - ik_ix^i)} = e^{\mp(ik_0t - ik_ix^i)} \Longrightarrow i\partial_te^{\mp(ik_0t - ik_ix^i)} = \hat{H}\Psi \Longrightarrow \boxed{\pm k_0\Psi = \hat{H}\Psi = E\Psi} \Longrightarrow$ $\pm k_0$ stands for positive and negative Energy
Dirac Equation Ansatz	$(i\partial\!\!\!/ - \mathbb{1}m)\Psi_{\alpha}(x^{\sigma}) = 0 \Longrightarrow (i\gamma^{\mu}\partial_{\mu} - \mathbb{1}m)\Psi_{\alpha}(x^{\sigma}) = 0$ ... (1) Ansatz: $\boxed{\Psi_{\alpha}^{(+)}(x^{\sigma}) = e^{-ik_{\mu}x^{\mu}}u_{\alpha}(k^{\sigma}) \text{ ... (2a); } \Psi_{\alpha}^{(-)}(x^{\sigma}) = e^{+ik_{\mu}x^{\mu}}v_{\alpha}(k^{\sigma}) \text{ ... (2b)}}$ $\Psi_{\alpha}^{(+)}$ ... pos. Energy sol., $\Psi_{\alpha}^{(-)}$ ... pos. Energy sol.
Condition positive Energy solution	$(i\gamma^{\mu}\partial_{\mu} - \mathbb{1}m)\Psi_{\alpha}^{(+)}(x^{\sigma}) = 0 \stackrel{(2a)}{\Longrightarrow} (i\gamma^{\mu}\partial_{\mu} - \mathbb{1}m)e^{-ik_{\mu}x^{\mu}}u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow (i\gamma^{\mu}(-ik_{\mu}) - \mathbb{1}m)e^{-ik_{\mu}x^{\mu}}u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow$ $e^{-ik_{\mu}x^{\mu}}(\gamma^{\mu}k_{\mu} - \mathbb{1}m)u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow e^{-ik_{\mu}x^{\mu}}(k\!\!\!/ - \mathbb{1}m)u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow \boxed{(k\!\!\!/ - \mathbb{1}m)u_{\alpha}(k^{\sigma}) = 0 \text{ with } k\!\!\!/ = \gamma^{\mu}k_{\mu}}$ ... (3a)
Condition negative Energy solution	$(i\gamma^{\mu}\partial_{\mu} - \mathbb{1}m)\Psi_{\alpha}^{(-)}(x^{\sigma}) = 0 \stackrel{(2b)}{\Longrightarrow} (i\gamma^{\mu}\partial_{\mu} - \mathbb{1}m)e^{+ik_{\mu}x^{\mu}}u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow (-\gamma^{\mu}k_{\mu} - \mathbb{1}m)e^{+ik_{\mu}x^{\mu}}u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow$ $e^{+ik_{\mu}x^{\mu}}(\gamma^{\mu}k_{\mu} + \mathbb{1}m)u_{\alpha}(k^{\sigma}) = 0 \Longrightarrow e^{+ik_{\mu}x^{\mu}}(k\!\!\!/ + \mathbb{1}m)v_{\alpha}(k^{\sigma}) = 0 \Longrightarrow \boxed{(k\!\!\!/ + \mathbb{1}m)v_{\alpha}(k^{\sigma}) = 0 \text{ with } k\!\!\!/ = \gamma^{\mu}k_{\mu}}$ ... (3b)
Adjoint positive Energy equation	$(3a)^{\dagger} \Longrightarrow ((k\!\!\!/ - \mathbb{1}m)u_{\alpha}(k^{\sigma}))^{\dagger} = 0 \Longrightarrow ((\gamma^{\mu}k_{\mu} - \mathbb{1}m)u_{\alpha}(k^{\sigma}))^{\dagger} = 0 \Longrightarrow u_{\alpha}^{\dagger}(k^{\sigma})((\gamma^{\mu})^{\dagger}k_{\mu} - \mathbb{1}m) = 0 \vert (\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0 \Longrightarrow$ $u_{\alpha}^{\dagger}(k^{\sigma})(\gamma^0\gamma^{\mu}\gamma^0k_{\mu} - \mathbb{1}m) = 0 \vert \mathbb{1} = \gamma^0\gamma^0 \Longrightarrow u_{\alpha}^{\dagger}(k^{\sigma})(\gamma^0\gamma^{\mu}\gamma^0k_{\mu} - \gamma^0\gamma^0m) = 0 \Longrightarrow$ $u_{\alpha}^{\dagger}(k^{\sigma})\gamma^0(\gamma^{\mu}k_{\mu} - \mathbb{1}m)\gamma^0 = 0 \vert u_{\alpha}^{\dagger}(k^{\sigma})\gamma^0 \stackrel{\text{def}}{=} \bar{u}_{\alpha}(k^{\sigma}) \Longrightarrow \bar{u}_{\alpha}(k^{\sigma})(\gamma^{\mu}k_{\mu} - \mathbb{1}m)\cancel{\gamma^0} = 0 \Longrightarrow \boxed{\bar{u}_{\alpha}(k^{\sigma})(k\!\!\!/ - \mathbb{1}m) = 0}$ ... (4a)
Adjoint negative Energy equation	$(3b)^{\dagger} \Longrightarrow ((k\!\!\!/ + \mathbb{1}m)v_{\alpha}(k^{\sigma}))^{\dagger} = 0 \Longrightarrow ((\gamma^{\mu}k_{\mu} + \mathbb{1}m)v_{\alpha}(k^{\sigma}))^{\dagger} = 0 \Longrightarrow v_{\alpha}^{\dagger}(k^{\sigma})((\gamma^{\mu})^{\dagger}k_{\mu} + \mathbb{1}m) = 0 \vert (\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0 \Longrightarrow$ $v_{\alpha}^{\dagger}(k^{\sigma})(\gamma^0\gamma^{\mu}\gamma^0k_{\mu} + \mathbb{1}m) = 0 \vert \mathbb{1} = \gamma^0\gamma^0 \Longrightarrow v_{\alpha}^{\dagger}(k^{\sigma})(\gamma^0\gamma^{\mu}\gamma^0k_{\mu} + \gamma^0\gamma^0m) = 0 \Longrightarrow$ $v_{\alpha}^{\dagger}(k^{\sigma})\gamma^0(\gamma^{\mu}k_{\mu} + \mathbb{1}m)\gamma^0 = 0 \vert v_{\alpha}^{\dagger}(k^{\sigma})\gamma^0 \stackrel{\text{def}}{=} \bar{v}_{\alpha}(k^{\sigma}) \Longrightarrow \bar{v}_{\alpha}(k^{\sigma})(\gamma^{\mu}k_{\mu} + \mathbb{1}m)\cancel{\gamma^0} = 0 \Longrightarrow \boxed{\bar{v}_{\alpha}(k^{\sigma})(k\!\!\!/ + \mathbb{1}m) = 0}$ ... (4b)
Rest-frame	$k^{\mu} = (m, \vec{0})^T, k_{\mu} = (m, \vec{0}) \stackrel{(3a)}{\Longrightarrow} (\gamma^0m - \mathbb{1}m)u_{\alpha}(m, \vec{0}) = 0 \vert : m \Longrightarrow \boxed{(\gamma^0 - \mathbb{1})u_{\alpha}(m, \vec{0}) = 0}$ ... (5a) $k^{\mu} = (m, \vec{0})^T, k_{\mu} = (m, \vec{0}) \stackrel{(3b)}{\Longrightarrow} (\gamma^0m + \mathbb{1}m)v_{\alpha}(m, \vec{0}) = 0 \vert : m \Longrightarrow \boxed{(\gamma^0 + \mathbb{1})v_{\alpha}(m, \vec{0}) = 0}$ ... (5b)

3.1 Solutions of the Dirac Equation in Dirac Representation (at rest)

Positive energy

$$(5a) \Longrightarrow \left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}-\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-2&0\\0&0&0&-2\end{pmatrix}\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow$$

$$-2\,u_3(m,\vec{0})=0,\ -2\,u_4(m,\vec{0})=0\Longrightarrow \boxed{u_\alpha(m,\vec{0})=\begin{pmatrix}\varphi\\0\end{pmatrix}}$$ ... positive energy solution in Dirac representation

Negative energy

$$(5b) \Longrightarrow \left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}+\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\begin{pmatrix}v_1(m,\vec{0})\\v_2(m,\vec{0})\\v_3(m,\vec{0})\\v_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}v_1(m,\vec{0})\\v_2(m,\vec{0})\\v_3(m,\vec{0})\\v_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow$$

$$2\,v_1(m,\vec{0})=0,\ 2\,v_2(m,\vec{0})=0\Longrightarrow \boxed{v_\alpha(m,\vec{0})=\begin{pmatrix}0\\\chi\end{pmatrix}}$$ ... negative energy solution in Dirac representation

Basis

$$u^{(1)}(m,\vec{0})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u^{(2)}(m,\vec{0})=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ v^{(1)}(m,\vec{0})=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ v^{(2)}(m,\vec{0})=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Spin Operator

$$\tfrac{1}{2}\Sigma_3=\tfrac{1}{2}\sigma_3\oplus\sigma_3=\tfrac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\Longrightarrow\tfrac{1}{2}\Sigma_3u^{(1)}=\tfrac{1}{2}u^{(1)};\tfrac{1}{2}\Sigma_3u^{(2)}=-\tfrac{1}{2}u^{(2)};\ \tfrac{1}{2}\Sigma_3u^{(3)}=\tfrac{1}{2}u^{(3)};\ \tfrac{1}{2}\Sigma_3u^{(4)}=-\tfrac{1}{2}u^{(4)}$$

$\Longrightarrow$ upper two components: positive energy spin up / spin down solution

$\Longrightarrow$ lower two components: negative energy spin up / spin down solution

Solutions of the Dirac Equation in Chiral Representation (at rest)

Positive energy

$$(5a) \Longrightarrow \left(\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}-\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow\begin{pmatrix}-1&0&1&0\\0&-1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow$$

$$\begin{pmatrix}-u_1+u_3\\u_2+u_4\\u_1-u_3\\u_2-u_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow u_1=u_3,u_2=u_4\Longrightarrow\boxed{u_\alpha(m,\vec{0})=\begin{pmatrix}\varphi\\\varphi\end{pmatrix}}$$ ... positive energy solution in Chiral representation

Negative energy

$$(5b) \Longrightarrow \left(\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}+\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}u_1(m,\vec{0})\\u_2(m,\vec{0})\\u_3(m,\vec{0})\\u_4(m,\vec{0})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow$$

$$\begin{pmatrix}u_1+u_3\\u_2+u_4\\u_1+u_3\\u_2-u_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Longrightarrow u_1=-u_3,u_2=-u_4\Longrightarrow\boxed{v_\alpha(m,\vec{0})=\begin{pmatrix}\chi\\-\chi\end{pmatrix}}$$ ... negative energy solution in Chiral representation

Solutions of the Dirac Equation for $\vec{k}\neq 0$ (Dirac representation)

$(\not{k}-\mathbb{1}m)\cdot(\not{k}+\mathbb{1}m)=0$

$$(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)=(\gamma^\mu k_\mu-\mathbb{1}m)(\gamma^\nu k_\nu+\mathbb{1}m)=\gamma^\mu k_\mu\gamma^\nu k_\nu-\mathbb{1}m^2=\gamma^\mu\gamma^\nu k_\mu k_\nu-\mathbb{1}m^2$$

$$(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)=\tfrac{1}{2}(\gamma^\mu\gamma^\nu k_\mu k_\nu+\underbrace{\gamma^\mu\gamma^\nu k_\mu k_\nu}_{\mu\leftrightarrow\nu})-\mathbb{1}m^2=\tfrac{1}{2}(\gamma^\mu\gamma^\nu k_\mu k_\nu+\gamma^\nu\gamma^\mu k_\nu k_\mu)-\mathbb{1}m^2\,\Big|\,k_\nu k_\mu=k_\mu k_\nu\Longrightarrow$$

$$(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)=\tfrac{1}{2}(\gamma^\mu\gamma^\nu k_\mu k_\nu+\gamma^\nu\gamma^\mu k_\mu k_\nu)-\mathbb{1}m^2=\tfrac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)k_\mu k_\nu-\mathbb{1}m^2=\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]_+k_\mu k_\nu-\mathbb{1}m^2\Longrightarrow$$

$$(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)=\tfrac{1}{2}2g^{\mu\nu}k_\mu k_\nu-\mathbb{1}m^2=k_\mu k^\mu-\mathbb{1}m^2=k^2-m^2=m^2-m^2=0\Longrightarrow\boxed{(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)=0}$$

General solution in Dirac representa-tion

$$\boxed{u_\alpha(k^\sigma)\stackrel{\text{def}}{=}\tfrac{1}{N}(\not{k}+\mathbb{1}m)\,u_\alpha(m,\vec{0})}\stackrel{(3a)}{\Longrightarrow}\boxed{\tfrac{1}{N}(\not{k}-\mathbb{1}m)(\not{k}+\mathbb{1}m)\,u_\alpha(m,\vec{0})=0}\ ...\ (7)$$

$$u_\alpha(k^\sigma)=\tfrac{1}{N}(\not{k}+\mathbb{1}m)\,u_\alpha(m,\vec{0})=\tfrac{1}{N}(\gamma^\mu k_\mu+\mathbb{1}m)\,u_\alpha(m,\vec{0})=\tfrac{1}{N}(\gamma^0k_0+\gamma^ik_i+\mathbb{1}m)\,u_\alpha(m,\vec{0})\,\Big|\,k_0=p_0=E\Longrightarrow$$

$$u_\alpha(k^\sigma)=\tfrac{1}{N}(\gamma^0E+\gamma^ik_i+\mathbb{1}m)\,u_\alpha(m,\vec{0})=\tfrac{1}{N}\left(\begin{pmatrix}\mathbb{1}_2&\mathbb{0}_2\\\mathbb{0}_2&-\mathbb{1}_2\end{pmatrix}E+\begin{pmatrix}\mathbb{0}_2&\sigma^i\\\sigma^i&\mathbb{0}_2\end{pmatrix}k_i+\begin{pmatrix}\mathbb{1}_2&\mathbb{0}_2\\\mathbb{0}_2&\mathbb{1}_2\end{pmatrix}m\right)\begin{pmatrix}\varphi\\0\end{pmatrix}\Longrightarrow$$

$$\boxed{u_\alpha(k^\sigma)\stackrel{\text{def}}{=}\tfrac{1}{N}\begin{pmatrix}(E+m)\varphi\\\sigma^ik_i\varphi\end{pmatrix}}$$ dominated by upper components if $|\vec{k}|\ll m$

$$\boxed{v_\alpha(k^\sigma)\stackrel{\text{def}}{=}\tfrac{1}{N}\begin{pmatrix}\sigma^ik_i\chi\\(E+m)\chi\end{pmatrix}}$$ dominated by lower components if $|\vec{k}|\ll m$

Normalization

$$\bar{u}^{(a)}(k^\sigma)\,u^{(b)}(k^\sigma)=\ \delta^{ab}\ ...\ (8a)\qquad \bar{u}^{(a)}(k^\sigma)\,v^{(b)}(k^\sigma)=0\ ...\ (8c)$$

$$\bar{v}^{(a)}(k^\sigma)\,v^{(b)}(k^\sigma)=-\delta^{ab}\ ...\ (8b)\qquad \bar{v}^{(a)}(k^\sigma)\,u^{(b)}(k^\sigma)=0\ ...\ (8d)$$

Inserting $u_\alpha(k^\sigma)$ and $v_\alpha(k^\sigma)\Longrightarrow\boxed{N=\sqrt{2m(E+m)}}$

3.2 Klein's Paradox

Scenario	Plane wave travelling in x_1-direction hits potential $V(\vec{x}) = \begin{cases} 0 \dots x_1 < 0 \\ V \dots x_1 \geq 0 \end{cases}$ | $x_1 < 0$: | $\vec{k} = \begin{pmatrix}1\\0\\0\end{pmatrix}\sqrt{E^2 - m^2}$ with $E > m$ | assump: spin up | $u(m,\vec{0}) = u^{(1)}(m,\vec{0}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Big\} \varphi = \begin{pmatrix}1\\0\end{pmatrix}$
Pos. Energy Spinor	$u_\alpha(k^\sigma) = \frac{1}{N}\begin{pmatrix}(E+m)\varphi \\ \vec{\sigma}\cdot\vec{k}\varphi\end{pmatrix} = \frac{1}{N}\begin{pmatrix}(E+m)\varphi \\ \sigma_x k_x \varphi\end{pmatrix} = \frac{1}{N}\begin{pmatrix}(E+m)\begin{pmatrix}1\\0\end{pmatrix} \\ \begin{pmatrix}0&1\\1&0\end{pmatrix}k_1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \Longrightarrow$ $\boxed{u_\alpha(k^\sigma) = \frac{1}{N}\begin{pmatrix}E+m\\0\\0\\k_1\end{pmatrix} = a\begin{pmatrix}1\\0\\0\\\frac{k_1}{E+m}\end{pmatrix}}$
Incoming wave (from left)	$\Psi_{in} = u_\alpha(k^\sigma)\,e^{-ik_\mu x^\mu} = e^{-iEt+ik_1x^1+ik_2x^2+ik_3x^3}\Big\vert k_2 = k_3 = 0 \Longrightarrow \Psi_{in} = u_\alpha(k^\sigma)\,e^{-iEt+ik_1x^1} \Longrightarrow$ $\Psi_{in}(t=0) = u_\alpha(k^\sigma)\,e^{ik_1x^1} = ae^{ik_1x^1}\begin{pmatrix}1\\0\\0\\\frac{k_1}{E+m}\end{pmatrix}$
Reflected wave (from right)	$\Psi_{refl} = be^{-ik_1x^1}\underbrace{\begin{pmatrix}1\\0\\0\\\frac{-k_1}{E+m}\end{pmatrix}}_{spin\ up} + b'e^{-ik_1x^1}\underbrace{\begin{pmatrix}0\\1\\\frac{-k_1}{E+m}\\0\end{pmatrix}}_{spin\ down}$ | Trans-mitted wave | $\Psi_{trans} = de^{ik'x^1}\underbrace{\begin{pmatrix}1\\0\\0\\\frac{k'}{E'+m}\end{pmatrix}}_{spin\ up} + d'e^{ik'x^1}\underbrace{\begin{pmatrix}0\\1\\\frac{k'}{E'+m}\\0\end{pmatrix}}_{spin\ down}$ | $E' = E - V$ $E'^2 = (E-V)^2$ $E'^2 = k'^2 + m^2$ $\Longrightarrow k' = \sqrt{(E-V)^2 - m^2}$
Continuity at $x_1 = 0$	$\Psi_{in}(0) + \Psi_{refl}(0) = \Psi_{trans}(0) \Longrightarrow a\begin{pmatrix}1\\0\\0\\\frac{k_1}{E+m}\end{pmatrix} + b\begin{pmatrix}1\\0\\0\\\frac{-k_1}{E+m}\end{pmatrix} + b'\begin{pmatrix}0\\1\\\frac{-k_1}{E+m}\\0\end{pmatrix} = d\begin{pmatrix}1\\0\\0\\\frac{k'}{E'+m}\end{pmatrix} + d'\begin{pmatrix}0\\1\\\frac{k'}{E'+m}\\0\end{pmatrix} \Longrightarrow$ $b' = d', b'(<0) = d'(>0) \Longrightarrow b' = d' = 0 \Longrightarrow$ no spin flips! $\dots \Longrightarrow \frac{b}{a} = \frac{1-\rho}{1+\rho};\ \frac{d}{a} = \frac{2}{1+\rho}$ with $\rho \stackrel{\text{def}}{=} \frac{k'}{k}\frac{E+m}{E-V+m}$
$V < E - m$	$R = \frac{j_{refl}}{j_{in}} = \left(\frac{b}{a}\right)^2 = \left(\frac{1-\rho}{1+\rho}\right)^2 \quad T = 1 - R = \frac{4\rho}{(1+\rho)^2}$ for $V \to 0 \Longrightarrow k' \to k \to \sqrt{E^2 - m^2} \Longrightarrow \rho \to 1 \Longrightarrow R \to 0, T \to 1$ as expected
$V = E - m$	$E - V \to m \Longrightarrow k' \to 0 \Longrightarrow \rho \to 0 \Longrightarrow R \to 1, T \to 0$
$E - m < V < E$	$E - V < m \Longrightarrow k'$ imaginary $\Longrightarrow$ transmitted wave is decaying. | Penetr. length | $d = \frac{1}{\sqrt{m^2 - (V-E)^2}}$
$V = E$	$d = d_{min} = \frac{1}{m}$ ...Compton wave length (for electrons: $4 \cdot 10^{13}$m)
$E < V < E + m$	Penetration length d becomes larger again
$V > E + m$	$E - V < -m \Longrightarrow k'$ real $\Longrightarrow E - V + m < 0 \Longrightarrow \rho < 0 \Longrightarrow \boxed{R > 1, T < 0}$ More is reflected than came in! KLEIN'S PARADOX **Points towards creation of particle pairs! Single particle theory like Dirac theory is not sufficient!**

Localizability of a Wave Packet

Wave Packet	$\Psi(0,\vec{x}) = e^{-\frac{\vec{x}^2}{2D^2}}\omega_\alpha$ ... (1) with $\omega_\alpha \mathrel{\hat{=}} \begin{pmatrix}\varphi\\0\end{pmatrix}$ spinor. The smaller D the more localized in space. $\Psi(0,\vec{p}) = \int \Psi(0,\vec{x})\,e^{-ip_\mu x^\mu}\,d^3x = \int e^{-\frac{\vec{x}^2}{2D^2}}\omega_\alpha e^{-ip_\mu x^\mu}\,d^3x =$ FT of Gaussian $= (2\pi D^2)^{\frac{3}{2}} e^{-\frac{\vec{p}^2 D^2}{2}}\omega_\alpha$ large $D \to$ localized $\vec{p}$
General Solution	$\Psi(t,\vec{x}) = \frac{1}{(2\pi)^3}\frac{m}{E}\int\sum_{a=1}^{2}\left(b(p;a)\,u^{(a)}(p)\,e^{-ip_\mu x^\mu} + d^*(p;a)\,v^{(a)}(p)\,e^{ip_\mu x^\mu}\right)d^3p$ with $p \mathrel{\hat{=}} p^\mu = (p_0, \vec{p})$ $\frac{m}{E}$ ... for convenience; $u^{(a)}(p)$ ... plane wave pos. energy contributions; $v^{(a)}(p)$ ... plane wave neg. energy contributions $b(p;a), d^*(p;a)$ ... expansion coefficients (star at d^* only for convenience)
Calculating Expansion Coefficients	$\int\Psi(0,\vec{x})\,e^{-ip_ix^i}\,d^3x = \int\left[\frac{1}{(2\pi)^3}\frac{m}{E}\int\sum_{a=1}^{2}\left(b(p;a)\,u^{(a)}(p')\,e^{-ip'_ix^\mu} + d^*(p;a)\,v^{(a)}(p')\,e^{ip'_ix^\mu}\right)d^3p'\right]_{t=0} e^{-ip_ix^i}\,d^3x$ ... (1) $\int\Psi(0,\vec{x})\,e^{-i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{x})}\,d^3x = (2\pi)^2\,\delta(\vec{p} - \vec{p}') \stackrel{(1)}{\Rightarrow} \dots \Longrightarrow$ $\boxed{\int\Psi(0,\vec{x})\,e^{-ip_ix^i}\,d^3x = \frac{m}{E}\sum_{a=1}^{2}\left(b(p;a)\,u^{(a)}(p) + d^*(\tilde{p};a)\,v^{(a)}(\tilde{p})\right)}$ ... (2) with $\tilde{p} \mathrel{\hat{=}} \tilde{p}^\mu = (p_0, -\vec{p})$ Orthogonality conditions (assuming wave in x-direction): $u^{\dagger(a)}(\tilde{k})\,u^{(b)}(k) = \delta^{ab}\frac{1}{N^2}(E+m, k_x, 0, 0)\begin{pmatrix}E+m\\k_x\\0\\0\end{pmatrix} = \delta^{ab}\frac{(E+m)^2 + k_x^2}{2m(E+m)} = \delta^{ab}\frac{(E+m)^2 + E^2 - m^2}{2m(E+m)} = \delta^{ab}\frac{2E^2 + 2Em}{2m(E+m)} = \delta^{ab}\frac{E}{m} \Longrightarrow$ $\boxed{u^{\dagger(a)}(\tilde{k})\,u^{(b)}(k) = \delta^{ab}\frac{E}{m};\ \text{analogously: } v^{\dagger(a)}(\tilde{k})\,v^{(b)}(k) = \delta^{ab}\frac{E}{m};\ v^{\dagger(a)}(\tilde{k})\,u^{(a)}(k) = 0;\ u^{\dagger(a)}(\tilde{k})\,v^{(b)}(k)}$ ... (3) $(1) = (2)\vert u^{\dagger(a)}\cdot \stackrel{(3)}{\Rightarrow} \boxed{b(p;a) = (2\pi D^2)^{\frac{3}{2}} e^{-\vec{p}^2\frac{D^2}{2}}\,u^{\dagger(a)}(p)\,\omega_\alpha}$ | $(1) = (2)\vert v^{\dagger(a)}\cdot \stackrel{(3)}{\Rightarrow} \boxed{d^*(p;a) = (2\pi D^2)^{\frac{3}{2}} e^{-\vec{p}^2\frac{D^2}{2}}\,v^{\dagger(a)}(p)\,\omega_\alpha}$
low energy	If $D \gg \frac{1}{m}$ (low energy)$\Longrightarrow d^* \ll b \Longrightarrow \frac{\text{amplitude negative energy solutions}}{\text{amplitude positive energy solutions}} \approx \frac{d^*}{b} = \frac{\vert\vec{p}\vert}{E+m} \ll 1$ almost only positive energy contributions
high energy zitterbeweg.	If $D \leq \frac{1}{m}$ (high energy) $\vec{j} = \bar{\Psi}\vec{\gamma}\Psi = \Psi^\dagger\vec{\gamma}\Psi$ has e^{2iEt} interference term $\Longrightarrow$ e.g. $2 \cdot 10^{21} Hz$ electron zitterbewegung, relates to interference term between pos. energy solution and neg. energy solution of a single particle if localized at narrow space.

3.3 Electromagnetic Coupling

Relativistic Field Strength Tensor	$F^{\mu\nu}=-F^{\mu\nu}=\begin{pmatrix}0 & -E^1 & -E^2 & -E^3\\ E^1 & 0 & -B^3 & B^2\\ E^2 & B^3 & 0 & -B^1\\ E^3 & -B^2 & B^1 & 0\end{pmatrix}$ Four-current: $j^\mu=\begin{pmatrix}\rho\\ \vec{j}\end{pmatrix}$ Max-well $\left.\begin{matrix}\vec\nabla\cdot\vec E=\rho\\ \vec\nabla\times\vec B-\partial_t\vec E=\vec j\end{matrix}\right\}\partial_\mu F^{\mu\nu}=j^\nu$ $\left.\begin{matrix}\vec\nabla\cdot\vec B=0\\ \vec\nabla\times\vec E+\partial_t\vec B=0\end{matrix}\right\}\partial_\mu\tilde F^{\mu\nu}=0$ Dual Field Tensor: $\tilde F^{\mu\nu}\stackrel{\text{def}}{=}\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}=\left\{F^{\mu\nu}\text{ with }\begin{pmatrix}\vec E\to-\vec B\\ \vec B\to\vec E\end{pmatrix}\right\}$
4-potential solves hom. Maxwell eq.	$\left.\begin{matrix}A^\mu=\begin{pmatrix}\phi\\ \vec A\end{pmatrix}\\ F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\end{matrix}\right\}\Leftrightarrow\left\{\begin{matrix}\vec B=\vec\nabla\times\vec A\\ \vec E=\partial_t\vec A-\vec\nabla\phi\end{matrix}\right.$ Local gauge trafo: $A^\mu(x^\sigma)\to A^\mu(x^\sigma)+\delta A^\mu(x^\sigma)=A^\mu(x^\sigma)+\partial_\mu\Lambda(x^\sigma)$ U(1) gauge group leaves $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ unchanged
Dirac equation: coupling to EM field (min. subst.)	How to bring em current $j^\mu=e\bar\Psi\gamma^\mu\Psi$ into Dirac equation $(i\gamma^\mu\partial_\mu-\mathbb{1}m)\Psi=0$ ... (1) ? min. subst.: $\boxed{\partial_\mu\to\partial_\mu+ieA_\mu}\stackrel{(1)}{\Rightarrow}$ $\left(i\gamma^\mu\left(\partial_\mu+ie\,A_\mu(x^\sigma)\right)-\mathbb{1}m\right)\Psi(x^\sigma)=0\Longrightarrow\left(i\gamma^\mu\partial_\mu-e\gamma^\mu A_\mu(x^\sigma)-\mathbb{1}m\right)\Psi(x^\sigma)=0\Longrightarrow$ $\boxed{\left(i\partial\!\!\!/-e\,A\!\!\!/(x^\sigma)-\mathbb{1}m\right)\Psi(x^\sigma)=0\text{ with }\partial\!\!\!/\stackrel{\text{def}}{=}\gamma^\mu\partial_\mu\text{ and }A\!\!\!/\stackrel{\text{def}}{=}\gamma^\mu A_\mu}$ ... (2) we can compensate $A^\mu(x^\sigma)\to A^\mu(x^\sigma)+\partial_\mu\Lambda(x^\sigma)$ with $\boxed{\Psi(x^\sigma)\to e^{-\Lambda(x^\sigma)}\Psi(x^\sigma)}$ so that (2) is fulfilled
"creating" A^μ	Other way round: By demanding $\Psi(x^\sigma)\to e^{-\Lambda(x^\sigma)}\Psi(x^\sigma)$, we automatically need to introduce $A^\mu(x^\sigma)\to A^\mu(x^\sigma)+\partial_\mu\Lambda(x^\sigma)$
Dirac equation with em coupling	$\gamma^\mu=\begin{pmatrix}\underline{\underline{\beta}}\\ \underline{\underline{\beta}}\,\underline{\underline{\alpha}}_i\end{pmatrix},\ A^\mu=\begin{pmatrix}\phi\\ A^i\end{pmatrix},\ A_\mu=(\phi,-A^i),\ \partial_\mu=(\partial_0,\partial_i)=-i\hat p_\mu=-i(\hat p_0,-\hat p_i),\ \hat p^\mu=\begin{pmatrix}\hat p^0\\ \hat p^i\end{pmatrix},\ \hat p_\mu=(\hat p_0,-\hat p_i),$ $\left(i\underline{\underline{\beta}}\partial_0+i\underline{\underline{\beta}}\,\underline{\underline{\alpha}}_i\partial_i-e\underline{\underline{\beta}}\phi\mathbb{1}+e\underline{\underline{\beta}}\,\underline{\underline{\alpha}}_iA^i-\underline{\underline{\beta}}\,\underline{\underline{\beta}}m\right)\Psi(x^\sigma)=0\Longrightarrow\underline{\underline{\beta}}\left(i\partial_0-\underline{\underline{\alpha}}_i\hat p_i-e\phi\mathbb{1}+e\underline{\underline{\alpha}}_iA^i-\underline{\underline{\beta}}m\right)\Psi(x^\sigma)=0\Longrightarrow$ $\boxed{i\frac{\partial}{\partial t}\Psi(x^\sigma)=\underbrace{\left(\vec\alpha\cdot\vec p+\underline{\underline{\beta}}m\right)}_{\hat H_0}\Psi(x^\sigma)+\underbrace{\left(-e\vec\alpha\cdot\vec A+e\phi\mathbb{1}\right)}_{\hat H_{int}}\Psi(x^\sigma)}$ ... (3) $\hat H=\vec\alpha\cdot\vec p+\underline{\underline{\beta}}m-e\vec\alpha\cdot\vec A+eA^0\mathbb{1}\Longrightarrow\boxed{\hat H=\underline{\underline{\beta}}m+\vec\alpha\cdot(\vec p-e\vec A)+eA^0\mathbb{1}}$ ... (3b)

3.3.1 Nonrelativistic Limit

Pauli Equation (1)	$\Psi(t,\vec x)=e^{-imt}\begin{pmatrix}\varphi(t,\vec x)\\ \chi(t,\vec x)\end{pmatrix}\stackrel{(3)}{\Rightarrow}i\frac{\partial}{\partial t}\left(e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right)=\left(\vec\alpha\cdot\vec p+\underline{\underline{\beta}}m\right)e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+\left(-e\vec\alpha\cdot\vec A+e\phi\right)e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\Bigg\vert\cdot e^{imt}$ $ie^{imt}\left(\frac{\partial}{\partial t}e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+e^{-imt}\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}\right)=\left(\vec\alpha\cdot\vec p+\underline{\underline{\beta}}m\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+\left(-e\vec\alpha\cdot\vec A+eA^0\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\Longrightarrow$ $ie^{imt}\left(-ime^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+e^{-imt}\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}\right)=\left(\vec\alpha\cdot\vec p+\underline{\underline{\beta}}m\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+\left(-e\vec\alpha\cdot\vec A+eA^0\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\xrightarrow{\text{Dirac representation}}$ $m\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+i\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}=\begin{pmatrix}\mathbb{0} & \sigma_i\\ \sigma_i & \mathbb{0}\end{pmatrix}\hat p^i\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+\begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}\end{pmatrix}m\begin{pmatrix}\varphi\\ \chi\end{pmatrix}-e\begin{pmatrix}\mathbb{0} & \sigma_i\\ \sigma_i & \mathbb{0}\end{pmatrix}A^i\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+eA^0\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\Longrightarrow$ $\begin{pmatrix}m\varphi\\ m\chi\end{pmatrix}+i\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}=\begin{pmatrix}\mathbb{0} & \sigma_i\hat p^i\\ \sigma_i\hat p^i & \mathbb{0}\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+\begin{pmatrix}\mathbb{1}m & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}m\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}-\begin{pmatrix}\mathbb{0} & e\sigma_iA^i\\ e\sigma_iA^i & \mathbb{0}\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}+eA^0\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\Longrightarrow$ $\begin{pmatrix}m\varphi\\ m\chi\end{pmatrix}+i\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}=\begin{pmatrix}\sigma_i\hat p^i\chi\\ \sigma_i\hat p^i\varphi\end{pmatrix}+\begin{pmatrix}m\varphi\\ -m\chi\end{pmatrix}-\begin{pmatrix}e\sigma_iA^i\chi\\ e\sigma_iA^i\varphi\end{pmatrix}+\begin{pmatrix}eA^0\varphi\\ eA^0\chi\end{pmatrix}\Longrightarrow$ $\begin{pmatrix}m\varphi\\ m\chi\end{pmatrix}+i\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}=\begin{pmatrix}\vec\sigma\cdot\vec p\ \chi+m\varphi-e\,\vec\sigma\cdot\vec A\,\chi+eA^0\varphi\\ \vec\sigma\cdot\vec p\ \varphi-m\chi-e\,\vec\sigma\cdot\vec A\,\varphi+eA^0\chi\end{pmatrix}\Longrightarrow\begin{pmatrix}m\varphi\\ m\chi\end{pmatrix}+i\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix}=\begin{pmatrix}\vec\sigma\cdot(\vec p-e\vec A)\,\chi+m\varphi+eA^0\varphi\\ \vec\sigma\cdot(\vec p-e\vec A)\,\varphi-m\chi+eA^0\chi\end{pmatrix}\Longrightarrow$ $\left.\begin{matrix}\not{m}\varphi+i\frac{\partial\varphi}{\partial t}=\vec\sigma\cdot(\vec p-e\vec A)\,\chi+\not{m}\varphi+eA^0\varphi\\ m\chi+i\frac{\partial\chi}{\partial t}=\vec\sigma\cdot(\vec p-e\vec A)\,\varphi-m\chi+eA^0\chi\end{matrix}\right\}m\text{ dominates}\Longrightarrow\left.\begin{matrix}i\frac{\partial\varphi}{\partial t}=\vec\sigma\cdot(\vec p-e\vec A)\,\chi+eA^0\varphi\\ \underbrace{i\frac{\partial\chi}{\partial t}}_{\ll m\chi}=\vec\sigma\cdot(\vec p-e\vec A)\,\varphi+\underbrace{eA^0}_{\ll m}\chi-2m\chi\end{matrix}\right\}\frac{\vert\vec p\vert}{m},\frac{\vert\vec A\vert}{m}\ll1,\Longrightarrow$ $\left.\begin{matrix}i\frac{\partial\varphi}{\partial t}=\vec\sigma\cdot(\vec p-e\vec A)\,\chi+eA^0\varphi\\ \chi\approx\frac{1}{2m}\vec\sigma\cdot(\vec p-e\vec A)\,\varphi\ll\varphi\end{matrix}\right\}\Longrightarrow i\frac{\partial\varphi}{\partial t}=\vec\sigma\cdot(\vec p-e\vec A)\,\frac{1}{2m}\vec\sigma\cdot(\vec p-e\,\vec A)\,\varphi+eA^0\varphi\Longrightarrow$ $\boxed{i\frac{\partial\varphi}{\partial t}\approx\left(\frac{1}{2m}(\vec\sigma\cdot\vec\pi)^2+eA^0\right)\varphi\ \text{ with }\vec\pi\stackrel{\text{def}}{=}\vec p-e\,\vec A(\vec x)}$ ... (4) Pauli Equation (1): describes fermions for $\frac{\vert\vec p\vert}{m},\frac{\vert\vec A\vert}{m}\ll1$

Pauli Equation (2)	$(\vec{\sigma}\cdot\vec{\pi})^2=\sigma_i\pi_i\sigma_j\pi_j=\sigma_i\sigma_j\pi_i\pi_j \vert \sigma_i\sigma_j=\delta_{ij}+i\varepsilon_{ijk}\sigma_k \Longrightarrow (\vec{\sigma}\cdot\vec{\pi})^2=(\delta_{ij}+i\varepsilon_{ijk}\sigma_k)\pi_i\pi_j=\pi_i\pi_i+i\varepsilon_{ijk}\sigma_k\pi_i\pi_j \Longrightarrow$ $(\vec{\sigma}\cdot\vec{\pi})^2=\pi_i\pi_i+\frac{1}{2}i\varepsilon_{ijk}\sigma_k\pi_i\pi_j+\frac{1}{2}i\underbrace{\varepsilon_{ijk}\sigma_k\pi_i\pi_j}_{i\leftrightarrow j}=\pi_i\pi_i+\frac{1}{2}i\varepsilon_{ijk}\sigma_k\pi_i\pi_j+\frac{1}{2}i\varepsilon_{jik}\sigma_k\pi_j\pi_i=\pi_i\pi_i+\frac{1}{2}i\varepsilon_{ijk}\sigma_k\pi_i\pi_j-\frac{1}{2}i\varepsilon_{ijk}\sigma_k\pi_j\pi_i$ $(\vec{\sigma}\cdot\vec{\pi})^2=\pi_i\pi_i+\frac{1}{2}i\varepsilon_{ijk}\sigma_k(\pi_i\pi_j-\pi_j\pi_i)\Longrightarrow \boxed{(\vec{\sigma}\cdot\vec{\pi})^2=\pi_i\pi_i+\frac{1}{2}i\sigma_k\varepsilon_{ijk}[\pi_i,\pi_j]}$ ... (5) $\vec{\pi}=\vec{p}-e\vec{A}=-i\vec{\nabla}-e\vec{A}\Longrightarrow \pi_i=-i\partial_i-eA_i\Longrightarrow [\pi_i,\pi_j]\varphi=[(-i\partial_i-eA_i),(-i\partial_j-eA_j)]\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=\varepsilon_{ijk}\left((-i\partial_i-eA_i)(-i\partial_j-eA_j)-(-i\partial_j-eA_j)(-i\partial_i-eA_i)\right)\varphi\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=\varepsilon_{ijk}(-\cancel{\partial_i\partial_j}+ie\partial_iA_j+ieA_i\partial_j+\cancel{e^2A_iA_j}+\cancel{\partial_j\partial_i}-ie\partial_jA_i-ieA_j\partial_i-\cancel{e^2A_jA_i})\varphi\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi+ie\underbrace{\varepsilon_{ijk}(A_i\partial_j-\partial_jA_i)}_{i\leftrightarrow j}\varphi=ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi+ie\varepsilon_{jik}(A_j\partial_i-\partial_iA_j)\varphi\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi-ie\varepsilon_{ijk}(A_j\partial_i-\partial_iA_j)\varphi=ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi+ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=2ie\varepsilon_{ijk}(\partial_iA_j-A_j\partial_i)\varphi=2ie\varepsilon_{ijk}(\partial_i(A_j\varphi)-A_j\partial_i\varphi)=2ie\varepsilon_{ijk}\left((\partial_iA_j)\varphi+\cancel{A_j\partial_i\varphi-A_j\partial_i\varphi}\right)\Longrightarrow$ $\varepsilon_{ijk}[\pi_i,\pi_j]\varphi=2ie\varepsilon_{ijk}(\partial_iA_j)\varphi\Longrightarrow \varepsilon_{ijk}[\pi_i,\pi_j]=2ie\varepsilon_{ijk}(\partial_iA_j)\overset{(5)}{\Longrightarrow}$ $(\vec{\sigma}\cdot\vec{\pi})^2=\pi_i\pi_i+\frac{1}{2}i\sigma_k2ie\varepsilon_{ijk}(\partial_iA_j)\Longrightarrow(\vec{\sigma}\cdot\vec{\pi})^2=\vec{\pi}^2-\varepsilon_{ijk}(\partial_iA_j)\sigma_ke \vert \varepsilon_{ijk}(\partial_iA_j)=B_k\Longrightarrow$ $\boxed{(\vec{\sigma}\cdot\vec{\pi})^2=\vec{\pi}^2-e\vec{\sigma}\cdot\vec{B}=(\vec{p}-e\vec{A})^2-e\vec{\sigma}\cdot\vec{B}}\overset{(4)}{\Longrightarrow} i\frac{\partial\varphi}{\partial t}=\left(\frac{1}{2m}\left((\vec{p}-e\vec{A})^2-e\vec{\sigma}\cdot\vec{B}\right)+eA^0\right)\varphi\Longrightarrow$ $\boxed{i\frac{\partial\varphi}{\partial t}=\left(\frac{1}{2m}(\vec{p}-e\vec{A})^2+eA^0-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right)\varphi}$ ... (6) ... Pauli Equation (2)
Potential for const. field	Claim: $\boxed{\vec{A}=\frac{1}{2}\vec{B}\times\vec{x}\text{ is the vector potential for }\vec{B}=const.}$ ... (7) Proof: $\vec{B}=\vec{\nabla}\times\vec{A} \vert \vec{A}=\frac{1}{2}\vec{B}\times\vec{x}\Longrightarrow\vec{B}=\frac{1}{2}\vec{\nabla}\times(\vec{B}\times\vec{x})\Longrightarrow B_i=\frac{1}{2}\varepsilon_{ijk}\partial_j(\vec{B}\times\vec{x})_k=\frac{1}{2}\varepsilon_{ijk}\partial_j(\varepsilon_{klm}B_lx_m)=\frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}B_l\partial_jx_m\Longrightarrow$ $B_i=\frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}B_l\delta_{jm}=\frac{1}{2}(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})B_l\delta_{jm}=\frac{1}{2}(\delta_{il}\delta_{jm}\delta_{jm}-\delta_{im}\delta_{jl}\delta_{jm})B_l=\frac{1}{2}(\delta_{il}\delta_{jj}-\delta_{im}\delta_{ml})B_l\Longrightarrow$ $B_i=\frac{1}{2}(\delta_{il}3-\delta_{il})B_l=\frac{1}{2}2\delta_{il}B_l\Longrightarrow B_i=B_i\,\blacksquare$
$(\vec{p}-e\vec{A})^2$ in constant small Field	$(\vec{p}-e\vec{A})^2\varphi=\left(\vec{p}-\frac{e}{2}\vec{B}\times\vec{x}\right)^2\varphi\mathrel{\widehat{=}}\left(\vec{p}-\frac{e}{2}\vec{B}\times\vec{x}\right)_i\left(\vec{p}-\frac{e}{2}\vec{B}\times\vec{x}\right)_i\varphi=\left(\hat{p}_i-\frac{e}{2}\varepsilon_{ijk}B_jx_k\right)\left(\hat{p}_i-\frac{e}{2}\varepsilon_{ilm}B_lx_m\right)\varphi\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}\hat{p}_i\hat{p}_i\varphi-\frac{e}{2}\hat{p}_i\varepsilon_{ilm}B_lx_m\varphi-\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i\varphi+O(B^2)\varphi \vert B_i=const\ll 1\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}\hat{p}_i\hat{p}_i\varphi-\frac{e}{2}\varepsilon_{ilm}B_l\hat{p}_ix_m\varphi-\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i\varphi\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}\hat{p}_i\hat{p}_i\varphi-\frac{e}{2}\varepsilon_{ilm}B_l(\hat{p}_ix_m+x_m\hat{p}_i)\varphi-\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i\varphi \vert \hat{p}_ix_m=-\partial_ix_m=-\delta_{im}\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}\hat{p}_i\hat{p}_i\varphi-\frac{e}{2}\varepsilon_{ilm}B_l(-\delta_{im}+x_m\hat{p}_i)\varphi-\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i\varphi\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}\hat{p}_i\hat{p}_i\varphi+\frac{e}{2}\varepsilon_{ilm}\delta_{im}B_l\varphi-\frac{e}{2}\varepsilon_{ilm}B_lx_m\hat{p}_i\varphi-\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i\varphi \vert \varepsilon_{ilm}\delta_{im}=0\Longrightarrow$ $(\vec{p}-e\vec{A})^2\varphi\mathrel{\widehat{=}}(\hat{p}_i\hat{p}_i-\frac{e}{2}\varepsilon_{ilm}B_lx_m\hat{p}_i-\underbrace{\frac{e}{2}\varepsilon_{ijk}B_jx_k\hat{p}_i}_{j\to l,k\to m})\varphi=\left(\hat{p}_i\hat{p}_i-\frac{e}{2}\varepsilon_{ilm}B_lx_m\hat{p}_i-\frac{e}{2}\varepsilon_{ilm}B_lx_m\hat{p}_i\right)\varphi=(\hat{p}_i\hat{p}_i-e\varepsilon_{ilm}B_lx_m\hat{p}_i)\varphi$ $\boxed{(\vec{p}-e\vec{A})^2\approx\hat{p}^2-e(\vec{x}\times\vec{p})\cdot\vec{B}=\hat{p}^2-e\vec{L}\cdot\vec{B}}$ ... (8) with angular momentum $\vec{L}=\vec{x}\times\vec{p}$
Pauli Hamiltonian	(8) in (6) $\Longrightarrow i\frac{\partial\varphi}{\partial t}\approx\left(\frac{1}{2m}(\hat{p}^2-e\vec{L}\cdot\vec{B})+eA^0-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right)\varphi=\left(\frac{1}{2m}\hat{p}^2-\frac{e}{2m}\vec{L}\cdot\vec{B}+eA^0-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right)\varphi$ $i\frac{\partial\varphi}{\partial t}\approx\left(\frac{1}{2m}\hat{p}^2+eA^0-\frac{e}{2m}(\vec{L}\cdot\vec{B}+\vec{\sigma}\cdot\vec{B})\right)\varphi \vert \vec{S}\overset{\text{def}}{=}\frac{1}{2}\vec{\sigma}\Longrightarrow\vec{\sigma}=2\vec{S}\Longrightarrow i\frac{\partial\varphi}{\partial t}\approx\left(\frac{1}{2m}\hat{p}^2+eA^0-\frac{e}{2m}(\vec{L}\cdot\vec{B}+2\vec{S}\cdot\vec{B})\right)\varphi\Longrightarrow$ $i\frac{\partial\varphi}{\partial t}\approx\left(\frac{1}{2m}\hat{p}^2+eA^0-\frac{e}{2m}(\vec{L}+2\vec{S})\cdot\vec{B}\right)\varphi\Longrightarrow\boxed{H_{Pauli}\approx\frac{1}{2m}\hat{p}^2+eA^0-\mu_B(\vec{L}+2\vec{S})\cdot\vec{B}}$ ... (9) with $\mu_B=\frac{e}{2m}$ Bohr magneton
gyrom. ratio	(9) $\Longrightarrow g=\frac{[magnetic\ moment]\cdot[spin]}{\mu_B}=2$ gyromagnetic ratio
relativistic spin-dependent interact.	$(i\not{\partial}-e\not{A}+\mathbb{1}m^2)$ acting on (2) $\Longrightarrow\cdots\Longrightarrow$ $\boxed{\left((i\not{\partial}-e\not{A})^2-\mathbb{1}m^2\right)\Psi=\left((i\partial-eA)^2-\mathbb{1}m^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\Psi=0}$ with $\sigma^{\mu\nu}\overset{\text{def}}{=}\frac{i}{2}[\gamma^\mu,\gamma^\nu]$ and $\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu})=(i\vec{\alpha}\cdot\vec{E}-\vec{\Sigma}\cdot\vec{B})$

3.4 Foldy-Wouthuysen Transformation

<table>
<tr><td>Aim</td><td>Systematically decoupling of large and small components of a Dirac spinor by (block)-diagonalizing the Hamiltonian</td></tr>
<tr><td>Ansatz</td><td>$\Psi' = e^{iS}\Psi \Leftrightarrow \Psi = e^{-iS}\Psi'$ ... (1) insert into Schrödinger-like equation $i\partial_t\Psi = \hat H\Psi \overset{(1)}{\Rightarrow} i\partial_t(e^{-iS}\Psi') = \hat H(e^{-iS}\Psi') \Longrightarrow$
$i(\partial_t e^{-iS})\Psi' + ie^{-iS}\partial_t\Psi' = \hat H(e^{-iS}\Psi') \,\big\vert\, e^{iS}\cdot \Longrightarrow i(\partial_t e^{-iS})\Psi' + ie^{-iS}\partial_t\Psi' = \hat H(e^{-iS}\Psi') \Longrightarrow$
$e^{iS}\left(i(\partial_t e^{-iS})\right)\Psi' + i\partial_t\Psi' = e^{iS}\hat H(e^{-iS}\Psi') \Longrightarrow i\partial_t\Psi' = e^{iS}\hat H(e^{-iS}\Psi') - e^{iS}\left(i(\partial_t e^{-iS})\right)\Psi' \Longrightarrow$
$i\partial_t\Psi' = e^{iS}\left(\hat H(e^{-iS}) - i(\partial_t e^{-iS})\right)\Psi' \Longrightarrow \boxed{\hat H' = e^{iS}\left(\hat H(e^{-iS}) - i(\partial_t e^{-iS})\right)}$ ... (2)</td></tr>
<tr><td>similar problm</td><td>Diagonalization of $\hat H = \sigma_x B_x + \sigma_z B_z$ achieved by rotation around y by $e^{\frac{i}{2}\sigma_y\Theta}$ with $\Theta = \arctan\left(\frac{B_x}{B_z}\right)$</td></tr>
</table>

3.4.1 Free Dirac Equation

<table>
<tr><td>Hamiltonian</td><td>$\hat H = \vec\alpha\cdot\vec p + \beta m$ Ansatz: $e^{iS} = \exp\left(\beta\vec\alpha\frac{\vec p}{\vert\vec p\vert}\Theta(p)\right) = \exp\left(\vec\gamma\frac{\vec p}{\vert\vec p\vert}\Theta(p)\right)$ ... (3)</td></tr>
<tr><td>$\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 = -\mathbb{1}$</td><td>$\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 \mathrel{\hat=} \frac{\gamma^i p_i\gamma^j p_j}{p_k p_k} = \frac{1}{p_kp_k}(\gamma^i\gamma^j p_ip_j) = \frac{1}{p_kp_k}\left(\frac12\gamma^i\gamma^jp_ip_j + \frac12\underbrace{\gamma^i\gamma^jp_ip_j}_{i\leftrightarrow j}\right) = \frac{1}{p_kp_k}\left(\frac12\gamma^i\gamma^jp_ip_j + \frac12\gamma^j\gamma^ip_jp_i\right) \Big\vert\, p_jp_i = p_ip_j \Longrightarrow$
$\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 \mathrel{\hat=} \frac{1}{p_kp_k}\left(\frac12\gamma^i\gamma^jp_ip_j + \frac12\gamma^j\gamma^ip_ip_j\right) = \frac{1}{p_kp_k}\frac12(\gamma^i\gamma^j+\gamma^j\gamma^i)p_ip_j$
$\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 \mathrel{\hat=} \frac{1}{p_kp_k}\frac12[\gamma^i,\gamma^j]_+p_ip_j \Big\vert\, \frac12[\gamma^i,\gamma^j]_+ = g^{\mu\nu}\mathbb{1} \Longrightarrow \frac12[\gamma^i,\gamma^j]_+ = -\delta_{ij}\mathbb{1} \Longrightarrow$
$\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 \mathrel{\hat=} -\frac{1}{p_kp_k}\delta_{ij}\mathbb{1}p_ip_j = -\frac{p_ip_i}{p_kp_k}\mathbb{1} \Longrightarrow \boxed{\left(\frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\right)^2 = -\mathbb{1}}$ ... (4)</td></tr>
<tr><td>expansion</td><td>$(4) \overset{(3)}{\Rightarrow}$ ... series ... $\Longrightarrow e^{iS} = \cos(\Theta) + \frac{\vec\gamma\cdot\vec p}{\vert\vec p\vert}\sin(\Theta) = \cos(\Theta) + \beta\frac{\vec\alpha\cdot\vec p}{\vert\vec p\vert}\sin(\Theta)$ ... (5)</td></tr>
<tr><td>Condition for diagonalizing $\hat H'$</td><td>Assumption: $\hat H$ and $\hat H'$ not time-dependent $\overset{(1)}{\Rightarrow} \hat H' = e^{iS}\hat He^{-iS} = e^{iS}\left(\vec\alpha\cdot\vec p + \beta m\right)e^{-iS} \overset{(4)}{\Rightarrow}$
$\hat H' = \left(\cos(\Theta) + \beta\frac{\vec\alpha\cdot\vec p}{\vert\vec p\vert}\sin(\Theta)\right)\left(\vec\alpha\cdot\vec p + \beta m\right)\left(\cos(\Theta) - \beta\frac{\vec\alpha\cdot\vec p}{\vert\vec p\vert}\sin(\Theta)\right)\Big\vert\, \beta\vec\alpha = -\vec\alpha\beta\vec\alpha,\ \beta\vec\alpha\beta = -\beta\beta\vec\alpha \Longrightarrow$
$\hat H' = \left(\vec\alpha\cdot\vec p + \beta m\right)\left(\cos(\Theta) - \beta\frac{\vec\alpha\cdot\vec p}{\vert\vec p\vert}\sin(\Theta)\right)^2 = \left(\vec\alpha\cdot\vec p + \beta m\right)\exp\left(-\beta\vec\alpha\frac{\vec p}{\vert\vec p\vert}\Theta\right)^2 = \left(\vec\alpha\cdot\vec p + \beta m\right)\exp\left(-2\beta\vec\alpha\frac{\vec p}{\vert\vec p\vert}\Theta\right) \Longrightarrow$
$\hat H' = \left(\vec\alpha\cdot\vec p + \beta m\right)\left(\cos(2\Theta) - \beta\frac{\vec\alpha\cdot\vec p}{\vert\vec p\vert}\sin(2\Theta)\right)$
$\hat H' = (\vec\alpha\cdot\vec p)\cos(2\Theta) - \beta\beta\frac{m}{\vert\vec p\vert}(\vec\alpha\cdot\vec p)\sin(2\Theta) + \beta m\cos(2\Theta) - \beta\frac{(\vec\alpha\cdot\vec p)^2}{\vert\vec p\vert}\sin(2\Theta)\Big\vert\, \beta\beta = \mathbb{1}, (\vec\alpha\cdot\vec p)^2 = -\vec p^2 \Longrightarrow$
$\hat H' = \vec\alpha\cdot\vec p\cos(2\Theta) - \frac{m}{\vert\vec p\vert}(\vec\alpha\cdot\vec p)\sin(2\Theta) + \beta m\cos(2\Theta) + \beta\frac{\vec p^2}{\vert\vec p\vert}\sin(2\Theta) \Longrightarrow$
$\hat H' = \vec\alpha\cdot\vec p\left(\cos(2\Theta) - \frac{m}{\vert\vec p\vert}\sin(2\Theta)\right) + \beta\left(m\cos(2\Theta) + \frac{\vec p^2}{\vert\vec p\vert}\sin(2\Theta)\right)$ ... (6) $\xrightarrow{\text{Dirac representation}}$
$\hat H' = \begin{pmatrix}\mathbb{0} & \sigma_i\\ \sigma_i & \mathbb{0}\end{pmatrix}\cdot\vec p\left(\cos(2\Theta) - \frac{m}{\vert\vec p\vert}\sin(2\Theta)\right) + \begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}\end{pmatrix}\left(m\cos(2\Theta) + \frac{\vec p^2}{\vert\vec p\vert}\sin(2\Theta)\right)\Big\vert$ want to get rid of off-diagonal terms
$\cos(2\Theta) - \frac{m}{\vert\vec p\vert}\sin(2\Theta) \overset{!}{=} 0 \Longrightarrow \cos(2\Theta) = \frac{m}{\vert\vec p\vert}\sin(2\Theta)\Big\vert :\cos(2\Theta) \Longrightarrow 1 = \frac{m}{\vert\vec p\vert}\frac{\sin(2\Theta)}{\cos(2\Theta)} = \frac{m}{\vert\vec p\vert}\tan(2\Theta) \Longrightarrow \boxed{\tan(2\Theta) = \frac{\vert\vec p\vert}{m}} \Longrightarrow$
$\frac{\sin(2\Theta)}{\cos(2\Theta)} = \frac{\vert\vec p\vert}{m} = \frac{\vert\vec p\vert}{E}\frac{E}{m} = \frac{\vert\vec p\vert/E}{m/E} \Longrightarrow \boxed{\sin(2\Theta) = \frac{\vert\vec p\vert}{E},\quad \cos(2\Theta) = \frac{m}{E}}$ ... (7)</td></tr>
<tr><td>Diagonalized Hamiltonian $\hat H'$</td><td>$(6) \Longrightarrow \hat H' = \vec\alpha\cdot\vec p\underbrace{\left(\cos(2\Theta) - \frac{m}{\vert\vec p\vert}\sin(2\Theta)\right)}_{\to 0} + \beta\left(m\cos(2\Theta) + \frac{\vec p}{\vert\vec p\vert}\sin(2\Theta)\right) \overset{(7)}{\Rightarrow}$
$\hat H' = \beta\left(\frac{m^2}{E} + \frac{\vec p^2}{\vert\vec p\vert}\frac{\vert\vec p\vert}{E}\right) = \beta\frac{m^2+\vec p^2}{E}\Big\vert\, E^2 = \vec p^2 + m^2 \Longrightarrow \hat H' = \beta E = \beta\sqrt{\vec p^2+m^2} = \begin{pmatrix}\mathbb{1} & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}\end{pmatrix}\sqrt{\vec p^2+m^2} \xrightarrow{\text{Dirac representation}}$
$\boxed{\hat H' = \begin{pmatrix}\mathbb{1}\sqrt{\vec p^2+m^2} & \mathbb{0}\\ \mathbb{0} & -\mathbb{1}\sqrt{\vec p^2+m^2}\end{pmatrix}}$</td></tr>
<tr><td>Trafo is non-local!</td><td>$\Psi'(x) = \langle x\vert\Psi'\rangle = \langle x\vert e^{iS}\Psi\rangle = \langle x\vert e^{iS}\mathbb{1}\Psi\rangle = \int\langle x\vert e^{iS}\vert x'\rangle\langle x'\vert\Psi\rangle\,dx' \Longrightarrow \boxed{\Psi'(x) = \int\langle x\vert e^{iS}\vert x'\rangle\,\Psi(x')\,dx'}$ non-local Trafo</td></tr>
<tr><td>Mean operator</td><td>The usual $\hat{\vec x}$ operator changes its meaning when applied to Ψ' and then transformed back, it becomes a "mean operator"
$e^{-iS}\hat{\vec x}\Psi' = e^{-iS}\hat{\vec x}e^{iS}\Psi = \hat{\vec x}_{mean}\Psi \Longrightarrow \boxed{\hat{\vec x}_{mean} = e^{-iS}\hat{\vec x}e^{iS}}$</td></tr>
</table>

3.4.2 Interacting Dirac Equation

Ansatz	$(3b)$ from 3.3 $\Longrightarrow \widehat{H} = \underline{\underline{\beta}} m + \vec{\alpha}\cdot(\vec{p} - e\vec{A}) + eA^0 \mathbb{1} \stackrel{!}{=} \underline{\underline{\beta}} m + \mathcal{E} + \mathcal{O}$ $\mathcal{O}$ ... off-diagonal ("odd") terms, $\mathcal{E}$ ... "even" terms
Free case (3.4.1)	$iS = \underline{\underline{\beta}}\mathcal{O}_{free}\frac{\Theta(p)}{\lvert\vec{p}\rvert} = \underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})\frac{\Theta(p)}{\lvert\vec{p}\rvert} \Longrightarrow \cdots \Longrightarrow \Theta = \frac{1}{2}\arctan\left(\frac{\lvert\vec{p}\rvert}{m}\right)$ for $\frac{\lvert\vec{p}\rvert}{m} \ll 1 \Longrightarrow \Theta \approx \frac{\lvert\vec{p}\rvert}{2m} \Longrightarrow iS \approx \underline{\underline{\beta}}\mathcal{O}_{free}\frac{1}{2m}$
ansatz interacting case	$\Psi' = e^{iS}\Psi$ with $e^{iS} = e^{\frac{1}{2m}\beta\mathcal{O}} \Longrightarrow \widehat{H}' = \underline{\underline{\beta}} m + \mathcal{E}' + \mathcal{O}'$ with $\mathcal{O}' \sim 0\left(\frac{1}{m}\right) \Longrightarrow$ $e^{iS'} = e^{\frac{1}{2m}\beta\mathcal{O}'} \Longrightarrow \widehat{H}'' = \underline{\underline{\beta}} m + \mathcal{E}'' + \mathcal{O}''$ with $\mathcal{O}'' \sim 0\left(\frac{1}{m^2}\right) \Longrightarrow e^{iS''} = e^{\frac{1}{2m}\beta\mathcal{O}''} \Longrightarrow \widehat{H}''' = \underline{\underline{\beta}} m + \mathcal{E}''' + \mathcal{O}''$ with $\mathcal{O}''' \sim 0\left(\frac{1}{m^3}\right)$ ...
Result	$\widehat{H}''' = \underline{\underline{\beta}} m + \mathcal{E}''' + 0\left(\frac{1}{m^3}\right)$ $\boxed{\widehat{H}''' = \underbrace{\underline{\underline{\beta}}\left(m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{\vec{p}^4}{8m^3}\right) + e\phi - \frac{e}{2m}\underline{\underline{\beta}}(\vec{\sigma}\cdot\vec{B})}_{(A)} + \underbrace{\left(-i\frac{e}{8m^2}\vec{\sigma}\cdot(\vec{\nabla}\times\vec{E}) - \frac{e}{4m^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p})\right)}_{(B)} - \underbrace{\frac{e}{8m^2}(\vec{\nabla}\cdot\vec{E})}_{(C)} + 0\left(\frac{1}{m^3}\right)}$ (A)... Pauli equation plus $\boxed{\text{relativistic correction } -\frac{\vec{p}^4}{8m^3}}$ coming from $E = \sqrt{m^2 + \vec{p}^2} \approx m + \frac{p^2}{2m} - \frac{\vec{p}^4}{8m^3} + \cdots$ (B)... $\boxed{\text{Spin} - \text{orbit interactions } \vec{p} \approx m\vec{v} \Longrightarrow -\frac{e}{4m^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p}) \approx -\frac{1}{2}\frac{e}{2m}\vec{\sigma}\cdot(\vec{E}\times\vec{v})}$ with $\vec{E}\times\vec{v}$ being the magnetic field as seen from the particle moving with velocity $\vec{v}$ through the electric field $\vec{E}$ $\Longrightarrow$Thomas procession. In a static, spherical symmetric potential the $\vec{\nabla}\times\vec{E}$ term vanishes, and in the second term $\vec{E}$ can be written as $\vec{E} = -\vec{\nabla}\phi = \frac{\vec{r}}{r}\cdot\frac{\partial\phi}{\partial r} \Longrightarrow$ $\boxed{\text{spin} - \text{orbit} - \text{term } (B) = \frac{e}{4m^2 r}\frac{d\phi}{dr}(\vec{L}\cdot\vec{\sigma}) \text{ with } \vec{L} = \vec{r}\times\vec{p}}$ (C)... Darwin term due to particle fluctuations ("zitterbewegung") $\langle\delta\vec{x}\rangle = 0, \langle(\delta\vec{x})^2\rangle > 0$

3.5 Hydrogen-like Atoms

3.5.1 Spinor Harmonics

$\left[\widehat{H}_0, \vec{\widehat{L}}\right] \neq \mathbb{0}$	$\left[\widehat{H}_0, \vec{\widehat{L}}\right] = \left[\vec{\alpha}\cdot\vec{p} + \underline{\underline{\beta}} m, \vec{x}\times\vec{p}\right] = \left[\vec{\alpha}\cdot\vec{p}, \vec{x}\times\vec{p}\right] + \left[\cancel{\underline{\underline{\beta}} m, \vec{x}\times\vec{p}}\right] = \vec{\alpha}\cdot\left[\vec{p}, \vec{x}\times(-i\vec{\nabla})\right] \mathrel{\widehat{=}} \alpha_i\left[\hat{p}_i, \varepsilon_{jkl}x_l\hat{p}_l\right]$ $\left[\widehat{H}_0, \vec{\widehat{L}}\right]_j \varphi = \alpha_i\varepsilon_{jkl}(\hat{p}_i x_k\hat{p}_l - x_k\hat{p}_l\hat{p}_i)\varphi = \alpha_i\varepsilon_{jkl}(\hat{p}_i(x_k\hat{p}_l\varphi) - x_k\hat{p}_l\hat{p}_i\varphi) = \alpha_i\varepsilon_{jkl}\left((\hat{p}_i x_k)\hat{p}_l\varphi + \cancel{x_k\hat{p}_i\hat{p}_l\varphi - x_k\hat{p}_l\hat{p}_i\varphi}\right)$ $\left[\widehat{H}_0, \vec{\widehat{L}}\right]_j = \alpha_i\varepsilon_{jkl}(-i\partial_i x_k)\hat{p}_l = -i\alpha_i\varepsilon_{jkl}\delta_{ik}\hat{p}_l \Longrightarrow \boxed{\left[\widehat{H}_0, \vec{\widehat{L}}\right]_j = -i\varepsilon_{jkl}\alpha_k\hat{p}_l = -i(\vec{\alpha}\times\vec{p})_j}$ ... (1)
$\left[\widehat{H}_0, \vec{\widehat{S}}\right] \neq \mathbb{0}$	$\left[\widehat{H}_0, \vec{\widehat{S}}\right]_i = \left[\vec{\alpha}\cdot\vec{p} + \underline{\underline{\beta}} m, \frac{1}{2}\Sigma\right]_i = \left[\alpha_j\hat{p}_j + \underline{\underline{\beta}} m, \frac{1}{2}\Sigma_i\right] = \left[\alpha_j\hat{p}_j + \underline{\underline{\beta}} m, \frac{i}{8}\varepsilon_{jkl}[\gamma^k, \gamma^l]\right] = \left[\alpha_j\hat{p}_j + \underline{\underline{\beta}} m, \frac{i}{8}\varepsilon_{jkl}\left[\underline{\underline{\beta}}\alpha_k, \underline{\underline{\beta}}\alpha_l\right]\right] = \cdots \Longrightarrow$ $\boxed{\left[\widehat{H}_0, \vec{\widehat{S}}\right]_i = i\varepsilon_{ikl}\alpha_k\hat{p}_l = +i(\vec{\alpha}\times\vec{p})_i}$ ... (2) $\Longrightarrow \boxed{\left[\widehat{H}_0, \Sigma\right] = 2\left[\widehat{H}_0, \vec{\widehat{S}}\right] = 2i(\vec{\alpha}\times\vec{p})}$ ... $(2b)$
$\left[\widehat{H}_0, \vec{J}\right] = \mathbb{0}$	$\left[\widehat{H}_0, \vec{J}\right] = \left[\widehat{H}_0, \vec{\widehat{L}} + \vec{\widehat{S}}\right] = \left[\widehat{H}_0, \vec{\widehat{L}}\right] + \left[\widehat{H}_0, \vec{\widehat{S}}\right] \stackrel{(1)(2)}{=} -i(\vec{\alpha}\times\vec{p}) + i(\vec{\alpha}\times\vec{p}) \Longrightarrow \boxed{\left[\widehat{H}_0, \vec{J}\right] = \mathbb{0}}$ ... (3) $\widehat{H}_0$ comm w. tot. ang momentum $\vec{J}$
spherical symmetric potential $A^0(r) \Longrightarrow$ $\left[\widehat{H}, \vec{J}\right] = \mathbb{0}$	$\boxed{\left[\vec{\Sigma}, f(r)\right] = \mathbb{0}}$ because $\vec{\Sigma}$ lives in spinor-space and $f(r)$ in physical space $\left[\vec{\widehat{L}}, f(r)\right]_i \varphi = \left[\vec{x}\times\vec{p}, f(r)\right]_i \varphi = \left(\vec{x}\times\vec{p}\, f(r) - f(r)\,\vec{x}\times\vec{p}\right)_i \varphi = \varepsilon_{ijk}x_j\hat{p}_k(f(r)\,\varphi) - f(r)\,\varepsilon_{ijk}x_j\hat{p}_k\varphi \,\rvert\, \hat{p}_k = -i\partial_k$ $\left[\vec{\widehat{L}}, f(r)\right]_i \varphi = -i\varepsilon_{ijk}x_j\partial_k(f(r)\,\varphi) + i\varepsilon_{ijk}\, f(r)\, x_j\partial_k\varphi = -i\varepsilon_{ijk}x_j(\partial_k f(r))\varphi - i\varepsilon_{ijk}x_j\, f(r)\,\partial_k\varphi + i\varepsilon_{ijk}\, f(r)\, x_j\partial_k\varphi$ ... (4) $\partial_k f(r) = f'(r)\,\partial_k r = f'(r)\,\partial_k\sqrt{x_l x_l} = f'(r)\,\partial_k(x_l x_l)^{\frac{1}{2}} = f'(r)\frac{1}{2}(x_l x_l)^{-\frac{1}{2}}\partial_k(x_m x_m) = f'(r)\frac{1}{2}\frac{1}{\sqrt{x_l x_l}}\left((\partial_k x_m)x_m + x_m\partial_k x_m\right)$ $\partial_k f(r) = f'(r)\frac{1}{2}\frac{1}{\sqrt{x_l x_l}}(\delta_{km}x_m + x_m\delta_{km}) = f'(r)\frac{1}{2}\frac{1}{\sqrt{x_l x_l}}(x_k + x_k) = f'(r)\frac{1}{2}\frac{2x_k}{\sqrt{x_l x_l}} = f'(r)\frac{x_k}{\sqrt{x_l x_l}} = f'(r)\frac{x_k}{\lvert x\rvert} \stackrel{(4)}{\Longrightarrow}$ $\left[\vec{\widehat{L}}, f(r)\right]_i \varphi = -i\varepsilon_{ijk}x_j\, f'(r)\frac{x_k}{\lvert x\rvert}\varphi\, \cancel{- i\varepsilon_{ijk}x_j\, f(r)\,\partial_k\varphi + i\varepsilon_{ijk}\, f(r)\, x_j\partial_k\varphi} = -i\underbrace{\varepsilon_{ijk}}_{asym} f'(r)\frac{1}{\lvert x\rvert}\underbrace{x_j x_k}_{symm} \Longrightarrow$ $\boxed{\left[\vec{\widehat{L}}, f(r)\right] = 0}$ ... (5) $\Longrightarrow$ for $\boxed{\widehat{H} = \vec{\alpha}\cdot\vec{p} + \underline{\underline{\beta}} m + e\, A^0(r)\,\mathbb{1}} \Longrightarrow \left[\widehat{H}, \vec{J}\right] = \left[\vec{\alpha}\cdot\vec{p} + \underline{\underline{\beta}} m + e\, A^0(r)\,\mathbb{1}, \vec{J}\right] = \left[\widehat{H}_0 + e\, A^0(r)\,\mathbb{1}, \vec{J}\right]$ $\left[\widehat{H}, \vec{J}\right] = \left[\cancel{\widehat{H}_0, \vec{J}}\right] + \left[e\, A^0(r)\,\mathbb{1}, \vec{J}\right] \stackrel{(3)}{=} -\left[\vec{J}, e\, A^0(r)\right] = -\left[\vec{\widehat{L}} + \vec{\widehat{S}}, e\, A^0(r)\,\mathbb{1}\right] \stackrel{(5)}{=} -\left[\cancel{\vec{\widehat{L}}, e\, A^0(r)\,\mathbb{1}}\right] - \left[\vec{\widehat{S}}, e\, A^0(r)\,\mathbb{1}\right] \Longrightarrow$ $\left[\widehat{H}, \vec{J}\right] = \left[\frac{1}{2}\Sigma, e\, A^0(r)\mathbb{1}\right] = \mathbb{0}$ because $\vec{\Sigma}$ lives in spinor space and $A^0(r)$ in physical space $\Longrightarrow$ $\boxed{\left[\widehat{H}, \vec{J}\right] = 0 \text{ in spherical symmetrical potential } A^0(r)}$

Spin-Orbit Operator

Spin-Orbit Operator $\hat{K}$

$$\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=\left[\hat{H},\underline{\underline{\beta}}\right]\vec{\Sigma}\cdot\vec{J}+\underline{\underline{\beta}}\left[\hat{H},\vec{\Sigma}\cdot\vec{J}\right]=\left[\hat{H},\underline{\underline{\beta}}\right]\vec{\Sigma}\cdot\vec{J}+\underline{\underline{\beta}}[\hat{H},\vec{\Sigma}]\cdot\vec{J}\ \ldots(6)$$

$$\left[\hat{H},\underline{\underline{\beta}}\right]=\left[\vec{\alpha}\cdot\vec{p}+\underline{\underline{\beta}}m+e\,A^0(r),\underline{\underline{\beta}}\right]=\left[\vec{\alpha}\cdot\vec{p},\underline{\underline{\beta}}\right]=(\vec{\alpha}\cdot\vec{p})\underline{\underline{\beta}}-\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})=\left(\vec{\alpha}\underline{\underline{\beta}}\right)\cdot\vec{p}-\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})\Big|\,\vec{\alpha}\underline{\underline{\beta}}=-\underline{\underline{\beta}}\vec{\alpha}\Longrightarrow$$

$$\left[\hat{H},\underline{\underline{\beta}}\right]=-\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})-\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})=-2\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})\ \ldots(7)\overset{(6)}{\Rightarrow}$$

$$\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=-2\underline{\underline{\beta}}(\vec{\alpha}\cdot\vec{p})\left(\vec{\Sigma}\cdot\vec{J}\right)+\underline{\underline{\beta}}[\hat{H},\vec{\Sigma}]\cdot\vec{J}\overset{(2b)}{\Longrightarrow}\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=2\underline{\underline{\beta}}\left(-(\vec{\alpha}\cdot\vec{p})\left(\vec{\Sigma}\cdot\vec{J}\right)+i(\vec{\alpha}\times\vec{p})\cdot\vec{J}\right)\ \ldots(8)$$

$$(\vec{\alpha}\cdot\vec{p})(\vec{\Sigma}\cdot\vec{J})=(\gamma^5\vec{\Sigma}\cdot\vec{p})(\vec{\Sigma}\cdot\vec{J})\,\widehat{=}\,\gamma^5\Sigma_i\hat{p}_i\Sigma_jJ_j=\gamma^5\Sigma_i\Sigma_j\hat{p}_iJ_j=\gamma^5\left(\tfrac{1}{2}[\Sigma_i,\Sigma_j]_++\tfrac{1}{2}[\Sigma_i,\Sigma_j]\right)\hat{p}_iJ_j\Longrightarrow$$

$$(\vec{\alpha}\cdot\vec{p})\left(\vec{\Sigma}\cdot\vec{J}\right)\widehat{=}\,\gamma^5(\delta_{ij}+i\varepsilon_{ijk}\Sigma_k)\hat{p}_iJ_j=\gamma^5(\hat{p}_iJ_i+i\varepsilon_{ijk}\Sigma_k\hat{p}_iJ_j)\,\widehat{=}\,\gamma^5(\vec{p}\cdot\vec{J}+i(\vec{\Sigma}\cdot\vec{p})\cdot\vec{J})\Big|\,\vec{\alpha}=\gamma^5\vec{\Sigma}\Longrightarrow$$

$$(\vec{\alpha}\cdot\vec{p})(\vec{\Sigma}\cdot\vec{J})=\gamma^5\vec{p}\cdot\vec{J}+i(\vec{\alpha}\times\vec{p})\cdot\vec{J}\Longrightarrow-(\vec{\alpha}\cdot\vec{p})(\vec{\Sigma}\cdot\vec{J})+i(\vec{\alpha}\times\vec{p})\cdot\vec{J}=-\gamma^5\vec{p}\cdot\vec{J}\overset{(8)}{\Rightarrow}$$

$$\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=-2\underline{\underline{\beta}}\gamma^5\vec{p}\cdot\vec{J}\,\Big|\,\vec{J}=\vec{L}+\vec{S}=\vec{x}\times\vec{p}+\tfrac{1}{2}\vec{\Sigma}\Longrightarrow\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=-2\underline{\underline{\beta}}\gamma^5\left(\vec{p}\cdot(\vec{x}\times\vec{p})+\tfrac{1}{2}\vec{p}\cdot\vec{\Sigma}\right)\ \ldots(9)$$

$$\vec{p}\cdot(\vec{x}\times\vec{p})\varphi\,\widehat{=}\,\varepsilon_{ijk}\hat{p}_i(x_j\hat{p}_k\varphi)=\underbrace{\varepsilon_{ijk}}_{asym}\underbrace{(\hat{p}_ix_j)}_{sym}\hat{p}_k\varphi+\underbrace{\varepsilon_{ijk}}_{asym}x_j\underbrace{\hat{p}_i\hat{p}_k}_{sym}\varphi=0\Longrightarrow\vec{p}\cdot(\vec{x}\times\vec{p})=0\overset{(8)}{\Rightarrow}$$

$$\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=-\underline{\underline{\beta}}\gamma^5\vec{p}\cdot\vec{\Sigma}\,\Big|\,\gamma^5\vec{\Sigma}=\vec{\alpha}\Longrightarrow\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]=-\underline{\underline{\beta}}\vec{p}\cdot\vec{\alpha}\overset{(7)}{=}\tfrac{1}{2}\left[\hat{H},\underline{\underline{\beta}}\right]\Longrightarrow\left[\hat{H},\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\right]-\tfrac{1}{2}\left[\hat{H},\underline{\underline{\beta}}\right]=0$$

$$\hat{H}\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}-\underline{\underline{\beta}}\vec{\Sigma}\cdot\vec{J}\hat{H}-\tfrac{1}{2}\hat{H}\underline{\underline{\beta}}+\tfrac{1}{2}\underline{\underline{\beta}}\hat{H}=0\Longrightarrow\hat{H}\,\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{J}-\tfrac{1}{2}\right)-\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{J}-\tfrac{1}{2}\right)\hat{H}=0\Longrightarrow\boxed{\left[\hat{H},\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{J}-\tfrac{1}{2}\right)\right]=[\hat{H},\hat{K}]=0}$$

Alternative forms of $\hat{K}$

$$\boxed{\hat{K}=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{J}-\tfrac{1}{2}\right)}\,\Big|\,\vec{J}=\vec{L}+\vec{S}=\vec{L}+\tfrac{1}{2}\vec{\Sigma}\Longrightarrow\hat{K}=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\left(\vec{L}+\tfrac{1}{2}\vec{\Sigma}\right)-\tfrac{1}{2}\right)=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{L}+\tfrac{1}{2}\vec{\Sigma}\cdot\vec{\Sigma}-\tfrac{1}{2}\right)\Big|\,\vec{\Sigma}\cdot\vec{\Sigma}=\mathbb{3}\Longrightarrow$$

$$\boxed{\hat{K}=\underline{\underline{\beta}}(\vec{\Sigma}\cdot\vec{L}+1)}=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{L}+\tfrac{3}{4}+\tfrac{1}{4}+\vec{L}^2-\vec{L}^2\right)=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{L}+\tfrac{1}{4}\vec{\Sigma}\cdot\vec{\Sigma}+\tfrac{1}{4}+\vec{L}^2-\vec{L}^2\right)=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{L}+\tfrac{1}{4}\vec{\Sigma}\cdot\vec{\Sigma}+\tfrac{1}{4}+\vec{L}^2-\vec{L}^2\right)\Longrightarrow$$

$$\hat{K}=\underline{\underline{\beta}}\left(\left(\vec{L}^2+\vec{L}\cdot\vec{\Sigma}+\tfrac{1}{4}\vec{\Sigma}\cdot\vec{\Sigma}\right)-\vec{L}^2+\tfrac{1}{4}\right)=\underline{\underline{\beta}}\left(\left(\vec{L}+\tfrac{1}{2}\vec{\Sigma}\right)^2-\vec{L}^2+\tfrac{1}{4}\right)=\underline{\underline{\beta}}\left(\left(\vec{L}+\vec{S}\right)^2-\vec{L}^2+\tfrac{1}{4}\right)\Longrightarrow\boxed{\hat{K}=\underline{\underline{\beta}}\left(\vec{J}^2-\vec{L}^2+\tfrac{1}{4}\right)}$$

$$\hat{K}=\underline{\underline{\beta}}\left(\vec{\Sigma}\cdot\vec{J}-\tfrac{1}{2}\right)=\begin{pmatrix}\mathbb{1}&\mathbb{0}\\\mathbb{0}&-\mathbb{1}\end{pmatrix}\left(\begin{pmatrix}\sigma_i&\mathbb{0}\\\mathbb{0}&\sigma_i\end{pmatrix}\cdot\vec{J}-\tfrac{1}{2}\right)\Longrightarrow\boxed{\hat{K}=\begin{pmatrix}\mathbb{k}&\mathbb{0}\\\mathbb{0}&-\mathbb{k}\end{pmatrix}}$$

Eigenvalues of $\vec{J}^2, J_z, \hat{K}$

In spherical symmetrical potential: $\boxed{[\hat{H},\hat{K}]=0;\ [\hat{H},\vec{J}]=0\ \Longrightarrow[\hat{H},\vec{J}^2]=0\Longrightarrow[\hat{H},J_z]=0}$ construct eigenstates with $\vec{J}^2, J_z, \hat{K}$

Eigenvalues $\boxed{J_z\!: m,\ \vec{J}^2\!:\ j(j+1),\ \ \vec{L}^2\!:\ l(l+1),\ \ \vec{S}\!:\ \pm\tfrac{1}{2}}$

$$\hat{K}=\underline{\underline{\beta}}\left(\vec{J}^2-\vec{L}^2+\tfrac{1}{4}\right)\Longrightarrow\boxed{k=j(j+1)-l(l+1)+\tfrac{1}{4}}\,\Big|\,j=l+s=l\pm\tfrac{1}{2}\Longrightarrow\boxed{k=\begin{cases}j+\tfrac{1}{2}&\text{for } j=l+\tfrac{1}{2}\\-\left(j+\tfrac{1}{2}\right)&\text{for } j=l-\tfrac{1}{2}\end{cases}}$$

Eigenspinors

("spinor harmonics") characterized by $j, m, \mathrm{sign}(k)$ and are linear combination of products of the eigenvectors of σ_z $\quad\chi_+=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_-=\begin{pmatrix}0\\1\end{pmatrix}$ $\quad$ and the spherical harmonics $Y_{l,l_z}(\vartheta,\varphi)$

Normalized spinor harmonics

$$\varphi_{jm}^{(+)}(\vartheta,\varphi)=\frac{1}{\sqrt{2l+1}}\begin{pmatrix}\sqrt{l+m+\tfrac{1}{2}}\,Y_{l,m-\frac{1}{2}}(\vartheta,\varphi)\\\sqrt{l-m+\tfrac{1}{2}}\,Y_{l,m+\frac{1}{2}}(\vartheta,\varphi)\end{pmatrix}\ \ldots\text{ for } j=l+\tfrac{1}{2}$$

$$\varphi_{jm}^{(-)}(\vartheta,\varphi)=\frac{1}{\sqrt{2l+1}}\begin{pmatrix}\sqrt{l-m+\tfrac{1}{2}}\,Y_{l,m-\frac{1}{2}}(\vartheta,\varphi)\\-\sqrt{l+m+\tfrac{1}{2}}\,Y_{l,m+\frac{1}{2}}(\vartheta,\varphi)\end{pmatrix}\ \ldots\text{ for } j=l-\tfrac{1}{2}$$

For a given value of j, the spinor harmonics with different sign of k have opposite parity (l differs by 1) $\frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\varphi_{jm}^{(\pm)}=\varphi_{jm}^{(\mp)}$

Spherical Harmonics:

$$Y_{l,m}(\vartheta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\vartheta)\,e^{im\varphi}$$

Associated Legendre polynomials:

$$P_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

3.5.2 Separation of Variables

Problem	Solve $i\frac{\partial\Psi}{\partial t}=\hat H\Psi$ for $\hat H=\underbrace{\vec\alpha\cdot\hat{\vec p}}_{off-diag}+\underbrace{\underline{\beta}}_{diag}m+\underbrace{e\,A^0(r)\,\mathbb{1}}_{diag}$ ... (1)	Separation ansatz:	$\Psi=\frac{1}{r}\begin{pmatrix} i\,g(r)\,\varphi^{\kappa}_{jm}(\vartheta,\varphi)\\ f(r)\,\varphi^{-\kappa}_{jm}(\vartheta,\varphi)\end{pmatrix}$ ... (2) with $\kappa=\mathrm{sign}(k)$ (eigenvalue of $\hat K$)

Radial Equations

$$i\frac{\partial\Psi}{\partial t}=\hat H\Psi\Rightarrow E\Psi=\hat H\Psi\Rightarrow(E\mathbb{1}-\hat H)\Psi=0\overset{(1)(2)}{\Longrightarrow}\left(E\mathbb{1}-\vec\alpha\cdot\hat{\vec p}-\underline{\beta}m-e\,A^0(r)\,\mathbb{1}\right)\frac1r\begin{pmatrix} i\,g(r)\,\varphi^{\kappa}_{jm}\\ f(r)\,\varphi^{-\kappa}_{jm}\end{pmatrix}\xrightarrow{\text{Dirac representation}}$$

$$\left(E\begin{pmatrix}\mathbb{1}&\mathbb{0}\\ \mathbb{0}&\mathbb{1}\end{pmatrix}-\begin{pmatrix}\mathbb{0}&\sigma_i\\ \sigma_i&\mathbb{0}\end{pmatrix}\hat p^i-\begin{pmatrix}\mathbb{1}&\mathbb{0}\\ \mathbb{0}&-\mathbb{1}\end{pmatrix}m-e\,A^0(r)\begin{pmatrix}\mathbb{1}&\mathbb{0}\\ \mathbb{0}&\mathbb{1}\end{pmatrix}\right)\frac1r\begin{pmatrix} i\,g(r)\,\varphi^{\kappa}_{jm}\\ f(r)\,\varphi^{-\kappa}_{jm}\end{pmatrix}=0\Rightarrow$$

$$\left(\begin{pmatrix}\mathbb{1}E&\mathbb{0}\\ \mathbb{0}&\mathbb{1}E\end{pmatrix}-\begin{pmatrix}\mathbb{0}&\sigma_i\hat p^i\\ \sigma_i\hat p^i&\mathbb{0}\end{pmatrix}-\begin{pmatrix}\mathbb{1}m&\mathbb{0}\\ \mathbb{0}&-\mathbb{1}m\end{pmatrix}-\begin{pmatrix}\mathbb{1}e\,A^0(r)&\mathbb{0}\\ \mathbb{0}&\mathbb{1}e\,A^0(r)\end{pmatrix}\right)\begin{pmatrix}\frac ir g(r)\,\varphi^{\kappa}_{jm}\\ \frac1r f(r)\,\varphi^{-\kappa}_{jm}\end{pmatrix}=0\Rightarrow$$

$$\begin{pmatrix}\mathbb{1}(E-m-e\,A^0(r))&-\sigma_i\hat p^i\\ -\sigma_i\hat p^i&\mathbb{1}(E+m-e\,A^0(r))\end{pmatrix}\begin{pmatrix}\frac ir g(r)\,\varphi^{\kappa}_{jm}\\ \frac1r f(r)\,\varphi^{-\kappa}_{jm}\end{pmatrix}=0\Rightarrow$$

$$\left.\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)\,\varphi^{\kappa}_{jm}-\tfrac1r\sigma_i\hat p^i f(r)\,\varphi^{-\kappa}_{jm}=0\\ &-\sigma_i\hat p^i\tfrac ir g(r)\,\varphi^{\kappa}_{jm}+(E+m-e\,A^0(r))\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)\,\varphi^{\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec p})\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}=0\\ &(E+m-e\,A^0(r))\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec p})\tfrac ir g(r)\,\varphi^{\kappa}_{jm}=0\end{aligned}\right\}\text{ ... (3)}$$

$$\vec\sigma\cdot\hat{\vec p}=\mathbb{1}(\vec\sigma\cdot\hat{\vec p})\,\Big|\,(\vec\sigma\cdot\hat{\vec x})^2=\mathbb{1}_2\text{ with }\hat{\vec x}\overset{\text{def}}{=}\frac{\vec x}{|\vec x|}=\frac{\vec x}{r}\Rightarrow\vec\sigma\cdot\hat{\vec p}=(\vec\sigma\cdot\hat{\vec x})^2(\vec\sigma\cdot\hat{\vec p})=(\vec\sigma\cdot\hat{\vec x})(\vec\sigma\cdot\hat{\vec x})(\vec\sigma\cdot\hat{\vec p})\,\Big|\,\hat{\vec x}\overset{\text{def}}{=}\frac{\vec x}{|\vec x|}=\frac{\vec x}{r}\Rightarrow$$

$$\vec\sigma\cdot\hat{\vec p}=(\vec\sigma\cdot\hat{\vec x})\frac1r(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\text{ ... (4)}\quad(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})=\sigma_ix_i\sigma_j\hat p_j=\sigma_i\sigma_jx_i\hat p_j\,\Big|\,\sigma_i\sigma_j=\delta_{ij}\mathbb{1}+i\varepsilon_{ijk}\sigma_k\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\,\widehat{=}\,(\delta_{ij}+i\varepsilon_{ijk}\sigma_k)x_i\hat p_j=x_j\hat p_j+i\varepsilon_{ijk}\sigma_kx_i\hat p_j\,\widehat{=}\,\vec x\cdot\hat{\vec p}+i\vec\sigma\cdot(\vec x\times\hat{\vec p})\,\Big|\,\vec x\times\hat{\vec p}=\hat{\vec L}\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})=\vec x\cdot\hat{\vec p}+i\vec\sigma\cdot\hat{\vec L}\,\Big|\,\hat{\vec p}=-i\vec\nabla\Rightarrow(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\,f(r)=\left(-i\vec x\cdot\vec\nabla+i\vec\sigma\cdot\hat{\vec L}\right)f(r)=-i\vec x\cdot\vec\nabla f(r)+i\vec\sigma\cdot\hat{\vec L}\,f(r)$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})=-i(\vec x\cdot\vec\nabla)\,f(r)+i\vec\sigma\cdot\hat{\vec L}\,f(r)\,\Big|\,(\vec x\cdot\vec\nabla)\,f(r)=r\frac{\partial}{\partial r}f(r)\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\,f(r)=-ir\frac{\partial}{\partial r}f(r)+i\vec\sigma\cdot\hat{\vec L}\,f(r)\,\Big|\,\frac{\partial}{\partial r}(r\,f(r))=\left(\frac{\partial}{\partial r}r\right)f(r)+r\frac{\partial}{\partial r}f(r)\Rightarrow r\frac{\partial}{\partial r}f(r)=\frac{\partial}{\partial r}(r\,f(r))-\left(\frac{\partial}{\partial r}r\right)f(r)\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\,f(r)=-i\left(\frac{\partial}{\partial r}(r\,f(r))-\left(\frac{\partial}{\partial r}r\right)f(r)\right)+i\vec\sigma\cdot\hat{\vec L}\,f(r)\,\Big|\,\frac{\partial}{\partial r}r=1\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})\,f(r)=-i\frac{\partial}{\partial r}(r\,f(r))+i\,f(r)+i\vec\sigma\cdot\hat{\vec L}\,f(r)=-i\left(\frac{\partial}{\partial r}r-1-\vec\sigma\cdot\hat{\vec L}\right)f(r)=\frac1i\left(\frac{\partial}{\partial r}r-\left(1+\vec\sigma\cdot\hat{\vec L}\right)\right)\Rightarrow$$

$$(\vec\sigma\cdot\vec x)(\vec\sigma\cdot\hat{\vec p})=\frac1i\left(\frac{\partial}{\partial r}r-\left(1+\vec\sigma\cdot\hat{\vec L}\right)\right)\overset{(4)}{\Rightarrow}(\vec\sigma\cdot\hat{\vec p})=(\vec\sigma\cdot\hat{\vec x})\frac1{ir}\left(\frac{\partial}{\partial r}r-\left(1+\vec\sigma\cdot\hat{\vec L}\right)\right)=(\vec\sigma\cdot\hat{\vec x})\frac1{ir}\left(\frac{\partial}{\partial r}r-\hat K\right)\overset{(3)}{\Rightarrow}$$

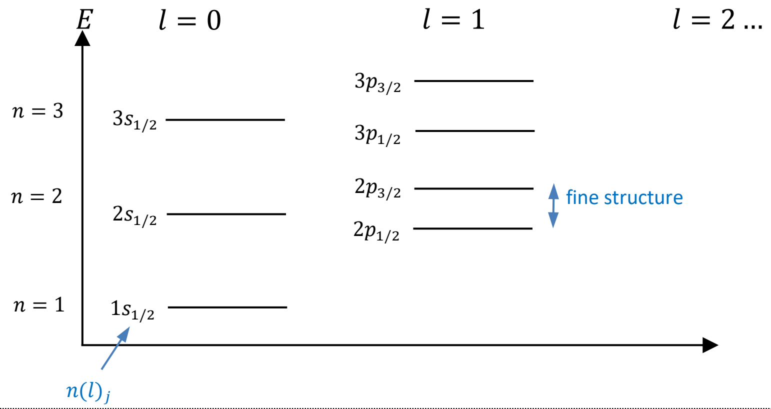
$$\left.\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)\,\varphi^{\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec x})\tfrac1{ir}\left(\tfrac{\partial}{\partial r}r-\hat K\right)\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}(\vartheta,\varphi)=0\\ &(E+m-e\,A^0(r))\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec x})\tfrac1{ir}\left(\tfrac{\partial}{\partial r}r-\hat K\right)\tfrac ir g(r)\,\varphi^{\kappa}_{jm}(\vartheta,\varphi)=0\end{aligned}\right\}\hat K\varphi^{\pm\kappa}_{jm}=k\varphi^{\pm\kappa}_{jm}\Rightarrow$$

$$\left.\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)\,\varphi^{\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec x})\tfrac1{ir}\varphi^{-\kappa}_{jm}\left(\tfrac{\partial}{\partial r}r+k\right)\tfrac{f(r)}{r}=0\\ &(E+m-e\,A^0(r))\tfrac1r f(r)\,\varphi^{-\kappa}_{jm}-(\vec\sigma\cdot\hat{\vec x})\tfrac1{ir}\varphi^{\kappa}_{jm}\left(\tfrac{\partial}{\partial r}r-k\right)\tfrac{i\,g(r)}{r}=0\end{aligned}\right\}(\vec\sigma\cdot\hat{\vec x})\frac1{ir}\varphi^{(\pm)}_{jm}=\frac1{ir}\varphi^{(\mp)}_{jm}\Rightarrow$$

$$\left.\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)\,\cancel{\varphi^{\kappa}_{jm}}-\tfrac1{ir}\cancel{\varphi^{\kappa}_{jm}}\left(\tfrac{\partial}{\partial r}r+k\right)\tfrac{f(r)}{r}=0\\ &(E+m-e\,A^0(r))\tfrac1r\,f(r)\,\cancel{\varphi^{-\kappa}_{jm}}-\tfrac1r\cancel{\varphi^{-\kappa}_{jm}}\left(\tfrac{\partial}{\partial r}r-k\right)\tfrac{g(r)}{r}=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}&(E-m-e\,A^0(r))\tfrac ir g(r)+\tfrac ir\left(\tfrac{\partial}{\partial r}+\tfrac kr\right)f(r)=0\,\Big|\cdot\tfrac ri\\ &(E+m-e\,A^0(r))\tfrac1r\,f(r)-\tfrac1r\left(\tfrac{\partial}{\partial r}-\tfrac kr\right)g(r)=0\,\Big|\cdot r\end{aligned}\right\}\Rightarrow$$

$$\boxed{\begin{aligned}&(E-m-e\,A^0(r))\,g(r)+\left(\tfrac{\partial}{\partial r}+\tfrac kr\right)f(r)=0\\ &(E+m-e\,A^0(r))\,f(r)-\left(\tfrac{\partial}{\partial r}-\tfrac kr\right)g(r)=0\end{aligned}}\text{ ... (5) radial equations}$$

3.5.3 Exact Solutions for the Coulomb Potential

Asymptotic behavior for $r \to \infty$	Coulomb Potential: $A^0 = -\frac{Ze}{4\pi r} \overset{(5)}{\Longrightarrow} \left.\begin{matrix}\left(E - m + \frac{Ze^2}{4\pi r}\right) g(r) + \left(\frac{\partial}{\partial r}+\frac{k}{r}\right) f(r) = 0 \\ \left(E+m+\frac{Ze^2}{4\pi r}\right) f(r) - \left(\frac{\partial}{\partial r}-\frac{k}{r}\right) g(r)=0\end{matrix}\right\} r\to\infty \Longrightarrow \begin{matrix}(E-m)\,g(r)+\frac{\partial f(r)}{\partial r}=0 \ldots (6a)\\ (E+m)\, f(r) - \frac{\partial g(r)}{\partial r} = 0 \ldots (6b)\end{matrix}$ $(6a) \Longrightarrow (E-m)\,g(r) = -\frac{\partial f(r)}{\partial r} \Longrightarrow g(r) = -\frac{1}{E-m}\frac{\partial f(r)}{\partial r} \Longrightarrow \frac{\partial g(r)}{\partial r} = -\frac{1}{E-m}\frac{\partial^2 f(r)}{\partial r^2} \overset{(6b)}{\Longrightarrow} (E+m)\, f(r) + \frac{1}{E-m}\frac{\partial^2 f(r)}{\partial r^2} = 0 \Big\vert (E-m)$ $(E+m)(E-m)\, f(r) + \frac{\partial^2 f(r)}{\partial r^2} = 0 \Longrightarrow \frac{\partial^2 f(r)}{\partial r^2} = -(E+m)(E-m)\, f(r) \Longrightarrow \frac{\partial^2 f(r)}{\partial r^2} = (m+E)(m-E)\, f(r) \Longrightarrow$ $\frac{\partial^2 f(r)}{\partial r^2} = (m^2-E^2)\, f(r) \xrightarrow{\text{characteristic polynomial}} \lambda^2 = m^2 - E^2 \Longrightarrow \lambda = \pm\sqrt{m^2-E^2}$ with ansatz $f(r) = Ae^{\lambda r} \Longrightarrow$ $\boxed{f(r\to\infty) = Ae^{\pm\sqrt{m^2-E^2}\, r}}$ ... (7) for $E < M$: damped solution, normalizable.
Generalized power series ansatz:	$f(r) = \sqrt{1-\frac{E}{m}}\, e^{-\lambda r}(F_1 - F_2)(\rho)$ ... (8a) $g(r) = \sqrt{1+\frac{E}{m}}\, e^{-\lambda r}(F_1-F_2)(\rho)$ ... (8b) with $\lambda \stackrel{\text{def}}{=} \sqrt{m^2-E^2}, \rho \stackrel{\text{def}}{=} 2\lambda r$ ... (8c) Power series: $F_{1,2} = \rho^\gamma(a_{1,2} + b_{1,2}\rho + c_{1,2}\rho^2+\cdots)$ ... (8d) $\overset{(5)}{\Longrightarrow} \ldots \Longrightarrow \gamma = \sqrt{k^2 - Z^2\alpha^2}$ ... (8e) with $\alpha = \frac{e^2}{4\pi}$ (fine struct. const)
Solutions	Degenerate hypergeometric functions $F(a,b;\rho) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{\rho^n}{n!}$ with $(a)_n = a(a+1)(a+2)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (pochhammer symbol) $(a)_n \le \infty$ iif $\boxed{a\le 0}$ $\rho^{-\gamma}F_1(\rho) = A\,F\left(\gamma+1-\frac{Z\alpha E}{\lambda}, 2\gamma+1;\rho\right)$ ... (9a) $\rho^{-\gamma}F_2(\rho) = B\,F\left(\gamma - \frac{Z\alpha E}{\lambda}, 2\gamma+1;\rho\right)$ ... (9b) $\frac{A}{B} = \frac{\gamma\lambda - Z\alpha E}{k\lambda + Z\alpha m}$ ... (9c) Radial quantum number: $n_r = -\left(\gamma - \frac{Z\alpha E}{\lambda}\right)$ ... (9d)
$n_r = 0$	If $n_r = 0$, condition $a\le 0$ (first parameter of F) is fulfilled in (9b), but not in (9a) $\Longrightarrow n_r = 0 \Longrightarrow A = 0$ $n_r = 0 \overset{(9d)}{\Longrightarrow} 0 = \gamma - \frac{Z\alpha E}{\lambda} \Longrightarrow \gamma = \frac{Z\alpha E}{\lambda} \Longrightarrow \gamma^2 = \frac{Z^2\alpha^2E^2}{\lambda^2} \overset{(8e)}{\Longrightarrow} k^2 - Z^2\alpha^2 = \frac{Z^2\alpha^2E^2}{\lambda^2} \overset{(8c)}{\Longrightarrow} k^2 - Z^2\alpha^2 = \frac{Z^2\alpha^2E^2}{m^2-E^2} \Longrightarrow$ $k^2 = \frac{Z^2\alpha^2E^2}{m^2-E^2} + Z^2\alpha^2 = \frac{Z^2\alpha^2(m^2-E^2)+Z^2\alpha^2E^2}{m^2-E^2} = Z^2\alpha^2\frac{m^2-E^2+E^2}{m^2-E^2} = Z^2\alpha^2\frac{m^2}{m^2-E^2} \overset{(8c)}{\Longrightarrow} k^2 = Z^2\alpha^2\frac{m^2}{\lambda^2} \Longrightarrow k(n_r=0) = \pm Z\alpha\frac{m}{\lambda}$ If we insert this into (9c), we see that the that the denominator gets zero. $\Longrightarrow k(n_r=0) = +Z\alpha\frac{m}{\lambda}$
Energies	$(9d) \Longrightarrow n_r = -\left(\gamma - \frac{Z\alpha E}{\lambda}\right) = -\gamma + \frac{Z\alpha E}{\lambda} \Longrightarrow \gamma + n_r = \frac{Z\alpha E}{\lambda} \Longrightarrow (\gamma+n_r)\lambda = Z\alpha E \Longrightarrow (\gamma+n_r)^2\lambda^2 = Z^2\alpha^2E^2 \overset{(8c)}{\Longrightarrow}$ $(\gamma+n_r)^2(m^2-E^2) = Z^2\alpha^2E^2 \Longrightarrow m^2 - E^2 = \frac{Z^2\alpha^2}{(\gamma+n_r)^2}E^2 \Longrightarrow \frac{Z^2\alpha^2}{(\gamma+n_r)^2}E^2 + E^2 = m^2 \Longrightarrow E^2\left(\frac{Z^2\alpha^2}{(\gamma+n_r)^2}+1\right) = m^2 \Longrightarrow$ $E = \frac{m}{\sqrt{1+\frac{Z^2\alpha^2}{(\gamma+n_r)^2}}} = m\left(1+\frac{Z^2\alpha^2}{(n_r+\gamma)^2}\right)^{-\frac12}$ ... (10) main quantum number $\boxed{n \stackrel{\text{def}}{=} n_r + \lvert k\rvert = n_r + j + \frac12}$ ... (11) $\Longrightarrow n_r = n - j - \frac12 \overset{(10)}{\Longrightarrow}$ $E = m\left(1+\frac{Z^2\alpha^2}{\left(n-(j+\frac12)+\gamma\right)^2}\right)^{-\frac12} \overset{(8e)}{\Longrightarrow} E = m\left(1+\frac{Z^2\alpha^2}{\left(n-(j+\frac12)+\sqrt{k^2-Z^2\alpha^2}\right)^2}\right)^{-\frac12} \Big\vert\ k^2 = \left(j+\frac12\right)^2 \Longrightarrow$ $\boxed{E = E_{nj} = m\left(1+\frac{Z^2\alpha^2}{\left(n-(j+\frac12)+\sqrt{(j+\frac12)^2 - Z^2\alpha^2}\right)^2}\right)^{-\frac12} = m\Big(1 - \underbrace{\frac{Z^2\alpha^2}{2n^2}}_{\text{Balmer spectrum}} - \underbrace{\frac{Z^4\alpha^4}{n^3(2j+1)} + \frac{3Z^4\alpha^4}{8n^4}}_{\text{fine structure}} + O(\alpha^6)\Big)}$ 
Singularities	Because $\gamma = \sqrt{k^2 - Z^2\alpha^2}$, γ becomes imaginary for $Z \ge 137 \Longrightarrow$ the wave function develops a n essential singularity at r=0 For $Z \ll 137$ and $\lvert k\rvert = 1, j = \frac12$ the wave function has a square integrable singularity at the origin. Only noticeable for $\ll 1$
Hyperfine structure	Caused by the coupling of the magnetic moment of the nucleus to the total angular momentum of the electron $\langle H_{hf}\rangle \propto \vec{\sigma}_e\cdot\vec{\sigma}_p \lvert\Psi(0)\rvert$ with $\Psi_{n,l=0}(0) = \sqrt{\frac{1}{\pi}\left(\frac{mZ\alpha}{n}\right)^3} \Longrightarrow \vec{\sigma}_e\cdot\vec{\sigma}_p = +1$ (triplett state) and -3 (singlet state) Only s-states are affected, because $\Psi_{n,l>0}(0) = 0$
Other effects	- Nuclear effects - two-body corrections: reduced mass $m^{-1} = m_e^{-1} + m_p^{-1}$ plus relativistic effects (recoil of nucleus) - radiative corrections (excited states are unstable, get a finite width due the possibility of the emission of photons)
Lamb Shift	Shift between $ns_{1/2}$ and $np_{1/2}$ states cannot be described with Dirac equation Estimate: $\Delta E_{lamb} = \frac{e}{6}\vec{\nabla}^2 A^0\langle(\delta x)^2\rangle \propto \frac{Z^4\alpha^5 m}{n^3}$ for $Z = 1, N = 2\ \approx$ 500 MHz (measured 1058 MHz)

4 Towards Many Body Theory

4.1 Hole Theory

Limits of Dirac Equation	Dirac equation is a one-particle wave theory. It describes the magnetic moment of the electron to some extent, and describes fine-structure of the Hydrogen atom. Problems: Klein's paradox, instability in the lowest energy eigenstates in the Coulomb potential. So far we ignored the negative energy solutions.
Neg. Energy Solutions	If we accept negative energy solutions, then there is no lowest energy level. Dirac proposes: All negative energies are already occupied by electrons (the "Dirac sea"). Also, by "lifting up" an electron with negative energy to positive energy level by inserting energy, I have created both an electron and a hole. The hole can be identified with the positron.
Vacuum polarizability	One electron at positive energy would slightly "push away" the electrons in the "Dirac sea". Hence, the electron polarizes the Dirac sea. A electron is therefore surrounded by a cloud of positive charge. The positive charges screen the electron on larger distances. At very small distances the charge of the electron becomes stronger. This is what happens. At low energy $\alpha \approx \frac{1}{137}$. At $m_w = 80\text{GeV}\ (\sim 10^{-3} fm)\ \alpha \approx \frac{1}{128}$
Problems with Hole Theory	Why is the Dirac sea filled with electrons and not, e.g., with positrons? Also, it relies on the Pauli principle, and therefore only works for fermions. But we also have the Klein-Gordon-Equation there one could have scalar particles, but for Bosonic particles the Pauli principle would not hold, and the explanation breaks down. The Hole theory is a historic artifact.

4.2 Charge Conjugation

electrons and positrons	Dirac equation for e^-: $(\mathrm{i}\not\partial - e\not A - \mathbb{1}m)\Psi = 0 \Longrightarrow (\mathrm{i}\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\mathbb{1})\Psi = 0 \Longrightarrow \gamma^\mu(\mathrm{i}\partial_\mu - eA_\mu - m\mathbb{1})\Psi = 0 \ldots (1a)$ Dirac equation for e^+: $(\mathrm{i}\not\partial + e\not A - \mathbb{1}m)\Psi^c = 0 \Longrightarrow (\mathrm{i}\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m\mathbb{1})\Psi^c = 0 \Longrightarrow \gamma^\mu(\mathrm{i}\partial_\mu + eA_\mu - m\mathbb{1})\Psi^c = 0 \ldots (1b)$
Charge conjugation	$(1a) \Longrightarrow (\mathrm{i}\not\partial - e\not A - m\mathbb{1})\Psi = 0 \Longrightarrow (\mathrm{i}\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\mathbb{1})\Psi = 0\big\vert^{\dagger} \Longrightarrow \Psi^\dagger(-\mathrm{i}(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu - m\mathbb{1}) = 0 \Longrightarrow$ $\Psi^\dagger((\gamma^\mu)^\dagger(-\mathrm{i}\overleftarrow{\partial}_\mu - eA_\mu) - m\mathbb{1}) = 0\big\vert(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \mathbb{1} = \gamma^0\gamma^0 \Longrightarrow \Psi^\dagger(\gamma^0\gamma^\mu\gamma^0(-\mathrm{i}\overleftarrow{\partial}_\mu - eA_\mu) - m\gamma^0\gamma^0) = 0 \Longrightarrow$ $\Psi^\dagger\gamma^0(\gamma^\mu(-\mathrm{i}\overleftarrow{\partial}_\mu - eA_\mu) - m)\cancel{\gamma^0} = 0\big\vert\Psi^\dagger\gamma^0 = \overline{\Psi} \Longrightarrow \overline{\Psi}(\gamma^\mu(-\mathrm{i}\overleftarrow{\partial}_\mu - eA_\mu) - m) = 0\big\vert^{T} \Longrightarrow$ $((\gamma^\mu)^T(-\mathrm{i}\partial_\mu - eA_\mu) - m)\overline{\Psi}^T = 0 \Longrightarrow ((-\gamma^\mu)^T(\mathrm{i}\partial_\mu - eA_\mu) - m)\mathbb{1}\overline{\Psi}^T = 0\big\vert$ introducing $\hat{C} \Longrightarrow \hat{C}^{-1}\hat{C} = \mathbb{1} \Longrightarrow$ $((-\gamma^\mu)^T(-\mathrm{i}\partial_\mu - eA_\mu) - m)\hat{C}^{-1}\hat{C}\overline{\Psi}^T = 0\big\vert\hat{C}\cdot \Longrightarrow \hat{C}((-\gamma^\mu)^T(\mathrm{i}\partial_\mu - eA_\mu) - m)\hat{C}^{-1}\hat{C}\overline{\Psi}^T = 0 \Longrightarrow$ $(\hat{C}(-\gamma^\mu)^T(\mathrm{i}\partial_\mu - eA_\mu)\hat{C}^{-1} - \hat{C}m\hat{C}^{-1})\hat{C}\overline{\Psi}^T = 0 \Longrightarrow (\hat{C}(-\gamma^\mu)^T\hat{C}^{-1}(\mathrm{i}\partial_\mu - eA_\mu) - \cancel{\hat{C}\hat{C}^{-1}}m)\hat{C}\overline{\Psi}^T = 0\big\vert$ compare with $(1b) \Longrightarrow$ $\boxed{\Psi^c = \hat{C}\overline{\Psi}^T} \ldots (2a)$ $\boxed{\gamma^\mu = \hat{C}(-\gamma^\mu)^T\hat{C}^{-1}} \ldots (2b)$
Dirac represent-tation	$\hat{C}_{dirac} = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ e.g. spin down spinor with negative energy and momentum in negative x-direction… $\Psi = e^{iEt-ikx}\begin{pmatrix} \frac{k}{e+m} \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\Psi^c = \hat{C}\overline{\Psi}^T = \hat{C}(\Psi^\dagger\gamma^0)^T = \hat{C}(\gamma^0)^T(\Psi^\dagger)^T = \hat{C}(\gamma^0)^T\Psi^* = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{k}{e+m} \\ 0 \\ 0 \\ 1 \end{pmatrix}e^{-iEt+ikx} \Longrightarrow$ $\Psi^c = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{k}{e+m} \\ 0 \\ 0 \\ -1 \end{pmatrix}e^{-iEt+ikx} \Longrightarrow \Psi^c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{k}{e+m} \end{pmatrix}$ …becomes a spin up spinor with positive energy and momentum in positive x-direction Charge conjugation can be viewed as a symmetry of the Dirac equation by combining $\Psi \to \Psi^c = \hat{C}\overline{\Psi}^T, A_\mu \to A_\mu^c = -A_\mu$
Chiral represent-tation	$\hat{C}_{chiral} = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ let $\Psi = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \Longrightarrow \Psi^c = \hat{C}(\gamma^0)^T\Psi^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix}\begin{pmatrix} \varphi^* \\ \vec{0} \end{pmatrix}$ $\Psi^c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \varphi_1^* \\ \varphi_2^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\varphi_2^* \\ \varphi_1^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ -i\sigma_2\varphi^* \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \varphi^c \end{pmatrix}$ …particle becomes anti-particle with opposite chirality. $\hat{C}$ is not a symmetry of chiral particles. But $\hat{C}\hat{P}$ is a symmetry of the Weyl eq. again.

5 Quantum Field Theory

5.1 Canonical Quantization

Action:	$I=\int_{t_1}^{t_2}\mathrm{L}(\mathrm{q}^i(t),\dot{\mathrm{q}}^i(t))\,dt$ Variation: $\mathrm{q}^i(t)\to\mathrm{q}^i(t)+\delta\mathrm{q}^i(t)$
Euler-Lagrange Equations	$\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}\delta q^i(t)+\frac{\partial L}{\partial\dot q^i(t)}\delta\dot q^i(t)\right)dt=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}\delta q^i(t)+\frac{\partial L}{\partial\dot q^i(t)}\frac{d}{dt}\delta q^i(t)\right)dt$ ... (1) $\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}\delta q^i(t)+\frac{\partial L}{\partial\dot q^i(t)}\delta\dot q^i(t)\right)dt=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}\delta q^i(t)+\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q^i(t)}\delta q^i(t)\right)-\left(\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}\right)\delta q^i(t)\right)dt$ $\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}-\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}\right)\delta\dot q^i(t)\,dt+\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q^i(t)}\delta q^i(t)\right)dt$ $\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}-\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}\right)\delta\dot q^i(t)\,dt+\left.\frac{\partial L}{\partial\dot q^i(t)}\delta q^i(t)\right\vert_{t_1}^{t_2}$ $\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}-\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}\right)\delta\dot q^i(t)\,dt+\frac{\partial L}{\partial\dot q^i(t)}(\delta q^i(t_2)-\delta q^i(t_1))$ $\vert$ $\delta q^i(t_2)-\delta q^i(t_1)=0$ $\delta I=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i(t)}-\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}\right)\delta\dot q^i(t)\,dt=0\Longrightarrow\boxed{\frac{\partial L}{\partial q^i(t)}-\frac{d}{dt}\frac{\partial L}{\partial\dot q^i(t)}=0}$ ... (1)
Conjugate momentum	$p_i=\frac{\partial L}{\partial\dot q^i}(q,\dot q)$ Hamiltonian: $\mathrm{H}(p,q)=p_i\dot q^i-\mathrm{L}(q,\dot q(p,q))$ Canonical quantization: $p_i(t)\to\hat p_i(t)$, $q^i(t)\to\hat q^i(t)$ Commutator relations: $[\hat q^i(t),\hat p_j(t)]=i\hbar\delta^i_j$

5.2 Quantization of the Free Complex Scalar Field (Klein Gordon Equation)

Concept	Instead of discrete coordinates the field has values at any point in space $\varphi(t,\vec x)\,\hat{=}\,\varphi_i(t)$ with $i=(\vec x,(\mathrm{Re},\mathrm{Im}))$ continuous
Action	for the Klein-Gordon equation $(\Box+m^2)\varphi=0$ $\boxed{I=\iint\mathcal{L}(\varphi,\partial\varphi)\,d^3x\,dt}$ ... (2) with $\boxed{\mathcal{L}=(\partial_\mu\varphi^*)(\partial^\mu\varphi)-m^2\varphi^*\varphi=(\partial_0\varphi^*)(\partial_0\varphi)-\vec\nabla\varphi^*\cdot\vec\nabla\varphi-m^2\varphi^*\varphi}$ ... (3)
Euler Lagrange	$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi}-\partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}=0}$ ... (4) analogous to (1). $\frac{\partial\mathcal{L}}{\partial\varphi}\overset{(3)}{=}-m^2\varphi^*$ ... (5a) $\frac{\partial\mathcal{L}}{\partial\varphi}\overset{(3)}{=}-m^2\varphi^*$ $\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\overset{(3)}{=}(\partial_\mu\varphi^*)$ ... (5b) $(5a),(5a)\overset{(4)}{\Rightarrow}-m^2\varphi^*-\partial^\mu(\partial_\mu\varphi^*)=0\;\vert\;\partial^\mu\partial_\mu\overset{\text{def}}{=}\Box\Longrightarrow\boxed{(\Box+m^2)\varphi^*=0}$ ...Klein-Gordon equation for complex conjugate φ^* To get the original Klein-Gordon equation we need to start with $\frac{\partial\mathcal{L}}{\partial\varphi^*}-\partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}=0$
Conjugate momentum	$\pi(t,\vec x)=\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi(t,\vec x))}$ ... (6) Hamiltonian: $\mathrm{H}=\int\boldsymbol{\mathcal{H}}(\pi,\varphi)\,d^3x=\int(\pi\partial_0\varphi+\pi^*\partial_0\varphi^*-\mathcal{L}(\varphi,\partial\varphi))\,d^3x$ ... (7)
Hamiltonian	$(3)\overset{(6)}{\Rightarrow}\left.\begin{array}{l}\pi=\partial_0\varphi^*=\dot\varphi^*\\ \pi^*=\partial_0\varphi=\dot\varphi\end{array}\right\}$ ... (8) $\overset{(3)}{\Rightarrow}\boxed{\mathcal{L}=\pi\pi^*-(\vec\nabla\varphi^*)(\vec\nabla\varphi)-m^2\varphi^*\varphi}\overset{(7)}{\Rightarrow}$ $\mathrm{H}=\int\left(\pi\pi^*+\pi^*\pi-\pi\pi^*+\vec\nabla\varphi^*\cdot\vec\nabla\varphi+m^2\varphi^*\varphi\right)d^3x\Longrightarrow\boxed{\mathrm{H}=\int\left(\pi^*\pi+\vec\nabla\varphi^*\cdot\vec\nabla\varphi+m^2\varphi^*\varphi\right)d^3x}$ ... (9)
Canonical quantization	$\varphi\to\hat\varphi,\ \varphi^*\to\hat\varphi^\dagger$, $\pi\to\hat\pi,\ \pi^*\to\hat\pi^\dagger$ Fields become operators! Commutator relations: $\boxed{[\hat\varphi(t,\vec x),\hat\pi(t,\vec y)]=[\hat\varphi^\dagger(t,\vec x),\hat\pi^\dagger(t,\vec y)]=i\hbar\,\delta^3(\vec x-\vec y)}$ ... (10)
Hamiltonian in momentum space	$\hat\pi(t,\vec x)\overset{FT}{=}\frac{1}{(2\pi)^3}\int e^{i\vec k\cdot\vec x}\,\tilde\pi(t,\vec k)\,d^3k$ $\hat\varphi(t,\vec x)\overset{FT}{=}\frac{1}{(2\pi)^3}\int e^{i\vec k\cdot\vec x}\,\tilde\varphi(t,\vec k)\,d^3k$ Insert into (9) and use that $\hat\pi^\dagger(t,\vec x)\overset{FT}{=}\frac{1}{(2\pi)^3}\int e^{-i\vec k'\cdot\vec x}\,\tilde\pi^\dagger(t,\vec k')\,d^3k'$ $\hat\varphi^\dagger(t,\vec x)\overset{FT}{=}\frac{1}{(2\pi)^3}\int e^{-i\vec k'\cdot\vec x}\,\tilde\varphi^\dagger(t,\vec k')\,d^3k'$ $\int e^{i\vec k\cdot\vec x}\,e^{-i\vec k'\cdot\vec x}\,d^3x=\delta^3(\vec k'-\vec k)\Longrightarrow$ $H=\frac{1}{(2\pi)^3}\int\left(\tilde\pi^*(t,\vec k)\,\tilde\pi(t,\vec k)+(-i\vec k)\,\varphi^*(t,\vec k)\,i\vec k\,\varphi(t,\vec k)+m^2\,\varphi^*(t,\vec k)\,\varphi(t,\vec k)\right)d^3k\Longrightarrow$ $H=\frac{1}{(2\pi)^3}\int\left(\tilde\pi^*(t,\vec k)\,\tilde\pi(t,\vec k)+\vec k^2\,\varphi^*(t,\vec k)\,\varphi(t,\vec k)+m^2\,\varphi^*(t,\vec k)\,\varphi(t,\vec k)\right)d^3k\Longrightarrow$ $H=\frac{1}{(2\pi)^3}\int\left(\lvert\tilde\pi(t,\vec k)\rvert^2+(\vec k^2+m^2)\lvert\varphi(t,\vec k)\rvert^2\right)d^3k\Longrightarrow$ $\boxed{H=\frac{1}{(2\pi)^3}\int\left(\lvert\tilde\pi(t,\vec k)\rvert^2+\omega_k^2\lvert\varphi(t,\vec k)\rvert^2\right)d^3k\ \text{ with }\ \omega_k^2\overset{\text{def}}{=}\vec k^2+m^2}$ ... (11) infinite number of harmonic oscillators
Harmonic oscillator 1D	creation operator: $\hat a^\dagger=\sqrt{\frac{m\omega}{2\hbar}}-\frac{1}{\sqrt2}\sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}$ anihilation operator: $\hat a=\sqrt{\frac{m\omega}{2\hbar}}+\frac{1}{\sqrt2}\sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}$ position operator: $\hat x=\sqrt{\frac{\hbar}{2m\omega}}(\hat a+\hat a^\dagger)$ Momentum operator: $\hat p=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat a-\hat a^\dagger)$ commutators: $[\hat a,\hat a^\dagger]=1,\ [\hat a,\hat a]=[\hat a^\dagger,\hat a^\dagger]=0,\ [\hat x,\hat p]=i\hbar$ Hamiltonian: $\hat H=\hbar\omega(\hat a\hat a^\dagger+\hat a^\dagger\hat a)=\hbar\omega\left(\hat a^\dagger\hat a+\frac12\right)$
Expressing fields with creation and annihilation operators	In our case we have a field and one harmonic oscillator at each spacetime point. We can write the fields as linear combination of creation and annihilation operators at every point in spacetime $\varphi(0,\vec x)=\frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat{\mathrm a}(\vec k)\,e^{i\vec k\cdot\vec x}+\hat{\mathrm b}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d^3k$ ... (12a) $\varphi^\dagger(0,\vec x)=\frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat{\mathrm b}(\vec k)\,e^{i\vec k\cdot\vec x}+\hat{\mathrm a}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d^3k$ ... (12b) $\pi(0,\vec x)=-\frac{i}{2}\frac{1}{(2\pi)^3}\int\left(\hat{\mathrm b}(\vec k)\,e^{i\vec k\cdot\vec x}-\hat{\mathrm a}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d^3k$ ... (12c) $\pi^\dagger(0,\vec x)=-\frac{i}{2}\frac{1}{(2\pi)^3}\int\left(\hat{\mathrm a}(\vec k)\,e^{i\vec k\cdot\vec x}-\hat{\mathrm b}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d^3k$ ... (12d)
Commutator	$(12abcd)\overset{(10)}{\Longrightarrow}\boxed{[\hat{\mathrm a}(\vec k),\hat{\mathrm a}^\dagger(\vec q)]=[\hat{\mathrm b}(\vec k),\hat{\mathrm b}^\dagger(\vec q)]=(2\pi)^3 2\omega_k\,\delta^3(\vec k-\vec q)}$ ... (13) Also with time dependency and four-vectors: $\boxed{[\hat{\mathrm a}(k^\mu),\hat{\mathrm a}^\dagger(q^\mu)]=[\hat{\mathrm b}(k^\mu),\hat{\mathrm b}^\dagger(q^\mu)]=(2\pi)^3 2\omega_k\,\delta^3(\vec k-\vec q)}$ ... (13b)
Lorentz invariance	$\boxed{\int d\tilde k\overset{\text{def}}{=}\int\frac{1}{(2\pi)^4}2\pi\,\delta(k^\mu k_\mu-m^2)\,\Theta(k^0)\,d^4k}$ ... (14a) is obviously inherently Lorentz invariant $\int d\tilde k=\frac{1}{(2\pi)^4}\int 2\pi\,\delta(k^\mu k_\mu-m^2)\,\Theta(k^0)\,d^4k=\frac{1}{(2\pi)^3}\int\delta(k_0^2-\vec k^2-m^2)\,\Theta(k^0)\,d^4k=\frac{1}{(2\pi)^3}\int\delta\left(k_0^2-(\vec k^2+m^2)\right)\Theta(k^0)\,d^4k$ $\int d\tilde k=\frac{1}{(2\pi)^3}\iint\delta(k_0^2-\omega_k^2)\,\Theta(k^0)\,dk_0\,d^3k$ $\vert$ $\delta(\mathrm f(k_0))=\frac{1}{\mathrm f'(k_0)}\delta(\mathrm f(k_0-\mathrm{root}(\mathrm f)))$ with f = $k_0^2-\omega_k^2$, root(f) = $+\omega_k$ bec. $\Theta(k_0)$ $\boxed{\int d\tilde k=\frac{1}{(2\pi)^3}\iint\frac{1}{2k_0}\delta(k_0-\omega_k)\,\Theta(k^0)\,dk_0 d^3k}$ ... (14b) $\Longrightarrow\boxed{\int d\tilde k=\frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}d^3k}$ ... (14c) also Lorentz invariant $\boxed{\varphi(0,\vec x)=\int\left(\hat{\mathrm a}(\vec k)\,e^{i\vec k\cdot\vec x}+\hat{\mathrm b}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d\tilde k}$ ... (15a) $\boxed{\varphi^\dagger(0,\vec x)=\int\left(\hat{\mathrm b}(\vec k)\,e^{i\vec k\cdot\vec x}+\hat{\mathrm a}^\dagger(\vec k)\,e^{-i\vec k\cdot\vec x}\right)d\tilde k}$ ... (15b)

Vacuum State and Particle Creation

Vacuum state	$\vert 0\rangle$ is the vacuum state, if $\forall k^\mu: \hat a(k^\mu)\vert 0\rangle = \hat b(k^\mu)\vert 0\rangle = 0$ and $\langle 0\vert 0\rangle = 1$
Hamiltonian	$\hat H = \frac{1}{2}\int \omega_k\left(\hat a^\dagger(k)\,\hat a(k) + \hat a(k)\,\hat a^\dagger(k) + \hat b^\dagger(k)\,\hat b(k) + \hat b(k)\,\hat b^\dagger(k)\right)d\tilde k$ $\quad\langle 0\vert\hat H\vert 0\rangle \to \infty$ because $\hat a^\dagger(k)\vert 0\rangle > 0$
Normal ord $\hat H$	$\hat H \to :\hat H: = \hat H - \langle 0\vert\hat H\vert 0\rangle = \int \omega_k\left(\hat a^\dagger(k)\,\hat a(k) + \hat b^\dagger(k)\,\hat b(k)\right)d\tilde k$ with :...: normal ordering, places all annihil. op's to the right
Particle creation	$\hat a^\dagger(k)\vert 0\rangle$ and $\hat b^\dagger(k)$ each create a particular particle with momentum $\vec k$ and energy ω_k. $\hat a^\dagger(k)\,\hat a^\dagger(k')\vert 0\rangle$ creates 2 particles. Because we started from the Klein-Gordon-Equation, we have Bose Statistics $[\hat a^\dagger(k), \hat a^\dagger(k')] = 0$
Fock Space	Union of all Hilbert sub-spaces with a given number of quanta n_a and n_b (eigenspaces of the number operators $\hat N_a$ and $\hat N_b$)
number oper.	$\hat N_a = \int \hat a^\dagger(k)\,\hat a(k)\,d\tilde k$ ... (15a) $\hat N_b = \int \hat b^\dagger(k)\,\hat b(k)\,d\tilde k$ ... (15b) (a and b corresponding to particles and anti-particles)
time evolution	Heisenberg picture: $\hat a(k,t) = e^{i\hat H t}\,\hat a(k)\,e^{-i\hat H t}$ ... (16) $[\hat H, \hat a(k)] = -\omega_k\,\hat a(k)$ ... (17) $\Longrightarrow$ $\hat a(k,t) = \hat a(k)\,e^{-i\omega_k t}$ ... (18) $\varphi(t,\vec x) \overset{(15a)}{=} \int\left(\hat a(\vec k,t)\,e^{i\vec k\cdot\vec x} + \hat b^\dagger(\vec k,t)\,e^{-i\vec k\cdot\vec x}\right)d\tilde k \overset{(18)}{\Longrightarrow} \varphi(t,\vec x) = \int\left(\hat a(k)\,e^{-i\omega_k t}e^{i\vec k\cdot\vec x} + \hat b^\dagger(k)\,e^{i\omega_k t}e^{-i\vec k\cdot\vec x}\right)d\tilde k$ $\varphi(t,\vec x) = \int\left(\hat a(k)\,e^{-i\omega_k t + i\vec k\cdot\vec x} + \hat b^\dagger(k)\,e^{i\omega_k t - i\vec k\cdot\vec x}\right)d\tilde k = \int\left(\hat a(k)\,e^{-i(\omega_k t - \vec k\cdot\vec x)} + \hat b^\dagger(k)\,e^{i(\omega_k t - \vec k\cdot\vec x)}\right)d\tilde k$ $\varphi(t,\vec x) = \frac{1}{(2\pi)^3}\iint\frac{1}{2k_0'}\left(\hat a(k)\,e^{-i(\omega_k t - \vec k\cdot\vec x)} + \hat b^\dagger(k)\,e^{i(\omega_k t - \vec k\cdot\vec x)}\right)\delta(k_0' - \omega_k)\,\Theta(k^0)\,dk_0'd^3k \Longrightarrow$ $\varphi(t,\vec x) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat a(k)\,e^{-i(\omega_k t - \vec k\cdot\vec x)} + \hat b^\dagger(k)\,e^{i(\omega_k t - \vec k\cdot\vec x)}\right)d^3k \;\big\vert\; \delta(k_0 - \omega_k)$ above meant we were picking $k_0 = \omega_k \Longrightarrow$ $\varphi(t,\vec x) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat a(k)\,e^{-i(k_0 t - \vec k\cdot\vec x)} + \hat b^\dagger(k)\,e^{i(k_0 t - \vec k\cdot\vec x)}\right)d^3k \;\big\vert\; t = x_0 \Longrightarrow$ $\varphi(t,\vec x) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat a(k)\,e^{-i(k_0 x_0 - \vec k\cdot\vec x)} + \hat b^\dagger(k)\,e^{i(k_0 x_0 - \vec k\cdot\vec x)}\right)d^3k \;\big\vert\; k_0 x_0 - \vec k\cdot\vec x = k_\mu x^\mu \Longrightarrow$ $\varphi(t,\vec x) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\left(\hat a(k)\,e^{-ik_\mu x^\mu} + \hat b^\dagger(k)\,e^{ik_\mu x^\mu}\right)d^3k \;\big\vert\; \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}d^3k = \int d\tilde k \Longrightarrow$ $\varphi(t,\vec x) = \int\left(\hat a(k)\,e^{-ik_\mu x^\mu} + \hat b^\dagger(k)\,e^{ik_\mu x^\mu}\right)d\tilde k$ ... (19a) $\varphi^\dagger(t,\vec x) = \int\left(\hat a^\dagger(k)\,e^{ik_\mu x^\mu} + \hat b(k)\,e^{-ik_\mu x^\mu}\right)d\tilde k$ ... (19b)

5.2.1 Causality

Same time	$[\varphi(t,\vec x), \varphi^\dagger(t,\vec x')] = 0$ .. two independent particles
Different times	$[\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] \overset{(19ab)}{=} \left[\int\left(\hat a(k)\,e^{-ik_\mu x^\mu} + \hat b^\dagger(k)\,e^{ik_\mu x^\mu}\right)d\tilde k, \int\left(\hat a^\dagger(k')\,e^{ik'_\mu x^{\mu'}} + \hat b(k)\,e^{-ik'_\mu x^{\mu'}}\right)d\tilde k'\right] \overset{(13)}{\Longrightarrow} \ldots \Longrightarrow$ $[\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] = \int\Big(\underbrace{e^{-ik\cdot(x-x')}}_{pos.energy} - \underbrace{e^{ik\cdot(x-x')}}_{neg.energy}\Big)d\tilde k = i\,\Delta(x^\mu - x^{\mu'})$ with Δ ... homogeneous Green function to GK-eq. Let $x - x' \overset{\text{def}}{=} z \Longrightarrow [\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] = \int\left(e^{-ik\cdot z} - e^{ik\cdot z}\right)d\tilde k = \int\left(e^{-i(k_0 z_0 + i\vec k\cdot\vec z)} - e^{i(k_0 z_0 + i\vec k\cdot\vec z)}\right)d\tilde k \Longrightarrow$ $[\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] = \int\frac{1}{(2\pi)^4}2\pi\,\delta\Big(\underbrace{k^\mu k_\mu - m^2}_{\underbrace{k_0^2 - \vec k^2 - m^2}_{-\omega_k^2}}\Big)\Theta(k^0)\left(e^{-i(k_0 z_0 + i\vec k\cdot\vec z)} - e^{i(k_0 z_0 + i\vec k\cdot\vec z)}\right)d^4k$ if $z_0 = t = 0 \to [\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] = 0$ when $\vec k \to -\vec k$ if $\vec z = 0$ and $z_0 = t \neq 0 \to [\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] \neq 0$ because of $\Theta(k^0)$ generally: $[\varphi(t,\vec x), \varphi^\dagger(t',\vec x')] = 0$ for all spatially separated regions (outside the light-cone particles are independent)$\Longrightarrow$ This preserves causality. This preservation of causality is only possible because of destructive interference between the positive energy and negative energy terms in the integral!

5.2.2 Internal Symmetry

Invariance	Klein-Gordon Lagrangian (3) is invariant under global U(1) transformations $\varphi \to \varphi' = e^{-i\alpha}\varphi,\quad \varphi^\dagger \to \varphi^{\dagger'} = e^{i\alpha}\varphi^\dagger$ ... (20) generated by $Q = \int\sum_a \pi_a(t,\vec x)\frac{\delta\varphi'_a(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0}d^3x = \int\left(\pi\frac{\delta\varphi'(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0} + \pi^\dagger\frac{\delta\varphi^{\dagger'}(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0}\right)d^3x$ ... (21) through commutators $[iQ\,\delta\alpha, \varphi_a] = \frac{\delta\varphi'_a(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0}\delta\alpha$ with $\varphi_a = \{\varphi, \varphi^\dagger\}$, $\varphi_a = \{\pi, \pi^\dagger\}$
Noether charge	$\dot Q = i[\hat H, Q] = 0$ ... Noether charge conserved *Let* $\varphi_a = \varphi \Longrightarrow \frac{\delta\varphi'(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0} = \frac{\delta}{\delta\alpha}\left(e^{-i\alpha}\varphi(t,\vec x)\right)\Big\vert_{\alpha=0} = -ie^{-i\alpha}\varphi(t,\vec x)\Big\vert_{\alpha=0} = -i\,\varphi(t,\vec x)$ *Let* $\varphi_a = \varphi^\dagger \Longrightarrow \frac{\delta\varphi^{\dagger'}(t,\vec x)}{\delta\alpha}\Big\vert_{\alpha=0} = \frac{\delta}{\delta\alpha}\left(e^{i\alpha}\varphi^\dagger(t,\vec x)\right)\Big\vert_{\alpha=0} = ie^{i\alpha}\varphi^\dagger(t,\vec x)\Big\vert_{\alpha=0} = i\,\varphi^\dagger(t,\vec x)$ $\;\overset{(21)}{\Longrightarrow}$ $Q = \int(-i\pi\varphi + i\pi^\dagger\varphi^\dagger)\,d^3x = i\int(\pi^\dagger\varphi^\dagger - \pi\varphi)\,d^3x \;\big\vert\; \pi = \dot\varphi^\dagger,\ \pi^\dagger = \dot\varphi \Longrightarrow Q = i\int(\varphi^\dagger\dot\varphi - \dot\varphi^\dagger\varphi)\,d^3x \to \infty \Longrightarrow$ $:Q: = \int\left(\hat a^\dagger\hat a - \hat b^\dagger\hat b\right)d\tilde k = \hat N_a - \hat N_b$ ... (22) total charge difference between particles and anti-particles

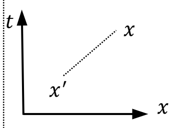
5.2.3 Time-Ordered Product and Feynman Propagator

charge transport

$(19a) \Longrightarrow \varphi(t,\vec{x}) = \int(\hat{a}\,... + \hat{b}^\dagger\,...)\,d\tilde{k}$ ... destroys a positive charge and/or creates a negative charge at $\vec{x}$

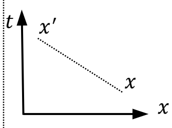
$(19b) \Longrightarrow \varphi^\dagger(t,\vec{x}) = \int(\hat{b}\,... + \hat{a}^\dagger\,...)\,d\tilde{k}$ ... destroys a negative charge and/or creates a positive charge at $\vec{x}$

positive charge transport $\vec{x}' \to \vec{x}$, $t' < t$



$x^{0\prime} < x^0 \Longrightarrow \varphi^\dagger(x^\mu)\,\varphi(x^{\mu\prime})$
read from right to left
first destroy pos. charge at $x^{\mu\prime}$
then create pos. charge at x^μ

positive charge transport $\vec{x} \to \vec{x}'$, $t < t'$



$x^0 < x^{0\prime} \Longrightarrow \varphi(x^{\mu\prime})\,\varphi^\dagger(x^\mu)$
read from right to left
first destroys neg. charge at x^μ
then create neg. charge at $x^{\mu\prime}$
(which is equivalent)

time-ordered product

$\boxed{\hat{T}\,\varphi(x')\,\varphi^\dagger(x) \stackrel{\text{def}}{=} \Theta(t'-t)\,\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,\varphi^\dagger(x)\,\varphi(x')}$ ... (23)

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\Theta(t'-t)\,\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,\varphi^\dagger(x)\,\varphi(x'))\,|\,\Theta(t'-t) = 1 - \Theta(t-t')$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)((1-\Theta(t-t'))\,\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,\varphi^\dagger(x)\,\varphi(x'))$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\varphi(x')\,\varphi^\dagger(x) - \Theta(t-t')\,\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,\varphi^\dagger(x)\,\varphi(x'))$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,(-\varphi(x')\,\varphi^\dagger(x) + \varphi^\dagger(x)\,\varphi(x')))$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\varphi(x')\,\varphi^\dagger(x) + \Theta(t-t')\,[\varphi^\dagger(x), \varphi(x')])$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\varphi(x')\,\varphi^\dagger(x)) + (\partial_{x'}^2 + m^2)(\Theta(t-t')\,[\varphi^\dagger(x), \varphi(x')])$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = (\partial_{x'}^2 + m^2)(\Theta(t-t')\,(\varphi^\dagger(x)\,\varphi(x') - \varphi(x')\,\varphi^\dagger(x)))$

...

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = \partial_{\mu,x'}\,\Theta(t-t')\,[\varphi^\dagger(x), \varphi(x')]$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = -\,\delta(t-t')\,[\varphi^\dagger(x), \partial^{\mu,x'}\varphi(x')]$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = -\,\delta(t-t')\,[\varphi^\dagger(x), \pi^\dagger(x')]$

$(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = -\,\delta(t-t')\,i\,\delta^3(\vec{x}-\vec{x}')$

$\boxed{(\Box_{x'} + m^2)\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = -i\,\delta^4(x'-x)}$ ... (24a) with x, x' ... four-vectors

also, without proof: $\boxed{\hat{T}\,\varphi(x')\,\varphi^\dagger(x) = \langle 0|\hat{T}\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle \; + \; :\varphi(x')\,\varphi^\dagger(x):}$

$\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = \langle 0|\int(\hat{a}(k)\,e^{-ik\cdot x'} + \hat{b}^\dagger(k)\,e^{ik\cdot x'})\,d\tilde{k}\int(\hat{a}^\dagger(q)\,e^{iq\cdot x'} + \hat{b}(q)\,e^{-iq\cdot x'})\,d\tilde{q}\,|0\rangle \Longrightarrow$

$\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = \iint\langle 0|\,\hat{a}(k)\,\hat{a}^\dagger(q)\,|0\rangle e^{-ik\cdot x'}\,e^{iq\cdot x'}\,d\tilde{k}\,d\tilde{q}$ ... (25)

$(13b) \Longrightarrow \hat{a}(k)\,\hat{a}^\dagger(q) - \hat{a}^\dagger(q)\,\hat{a}(k) = (2\pi)^3 2\omega_k\,\delta^3(\vec{k}-\vec{q}) \Longrightarrow \hat{a}(k)\,\hat{a}^\dagger(q) = (2\pi)^3 2\omega_k\,\delta^3(\vec{k}-\vec{q}) + \hat{a}^\dagger(q)\,\hat{a}(k) \stackrel{(25)}{\Longrightarrow}$

$\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = \iint(\langle 0|(2\pi)^3 2\omega_k\,\delta^3(\vec{k}-\vec{q})\,|0\rangle + \langle 0|\,\hat{a}^\dagger(q)\,\hat{a}(k)\,|0\rangle)e^{-ik\cdot x'}\,e^{iq\cdot x'}\,d\tilde{k}\,d\tilde{q} + \langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle \Longrightarrow$

$\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = (2\pi)^3\iint 2\omega_k\,\delta^3(\vec{k}-\vec{q})\,\langle 0|0\rangle e^{-ik\cdot(x'-x)}\,d\tilde{k}\,d\tilde{q}\,\Big|\,\int d\tilde{q} = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}\,d^3q \Longrightarrow$

$\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = \frac{(2\pi)^3}{(2\pi)^3}\iint\frac{2\omega_k}{2\omega_k}\,\delta^3(\vec{k}-\vec{q})\,e^{-ik\cdot(x'-x)}\,d\tilde{k}\,d\tilde{q} \Longrightarrow \boxed{\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle = \int e^{-ik\cdot(x'-x)}\,d\tilde{k}}$ ... (26)

Feynman Propagator $G_F(x-x')$ manifestly Lorentz invariant

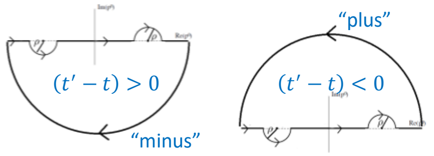
Feynman Propagator: $\boxed{G_F(x-x') = -\frac{1}{(2\pi)^4}\int\frac{1}{k^2-m^2+i\varepsilon}\,e^{-ik\cdot(x'-x)}\,d\tilde{k}}$ ... (27a) with x, x' ... four-vectors

$k^2 - m^2 + i\varepsilon = k_0^2 - \vec{k}^2 m^2 + i\varepsilon\,|\,k_0^2 \stackrel{\text{def}}{=} \omega^2 \Longrightarrow k^2 - m^2 + i\varepsilon = \omega^2 - (\vec{k}^2 + m^2) + i\varepsilon\,|\,\vec{k}^2 + m^2 = \omega_k^2 \Longrightarrow$

$k^2 - m^2 + i\varepsilon = \omega^2 - \omega_k^2 + i\varepsilon = (\omega - (\omega_k - i\tilde{\varepsilon}))(\omega - (-\omega_k - i\tilde{\varepsilon})) + O(\tilde{\varepsilon}^2) \stackrel{(26)}{\Longrightarrow}$

$\boxed{G_F(x-x') = -\frac{1}{(2\pi)^4}\int\frac{1}{\omega^2-\omega_k^2+i\varepsilon}\,e^{-ik\cdot(x'-x)}\,d\tilde{k} = -\frac{1}{(2\pi)^4}\int\frac{e^{-i\omega(t'-t)}\,e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}}{(\omega-(\omega_k-i\tilde{\varepsilon}))(\omega-(-\omega_k-i\tilde{\varepsilon}))}\,d\tilde{k}}$ ... (27b)

Solve with contour integral and residual theorem:
$\frac{1}{2\pi i}\int f(\omega)\,d\omega = \mathrm{Res}(f(\omega)) = \lim_{\omega\to\omega_0} f(\omega)$
(for a pole of first order)



$G_F(x-x') = -i\frac{1}{(2\pi)^3}\int\left(-\Theta(t'-t)\,\frac{e^{-i\omega_k(t'-t)}\,e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}}{2\omega_k} + \Theta(t'-t)\,\frac{e^{i\omega_k(t'-t)}\,e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}}{-2\omega_k}\right)d^3k\,\Big|\,\vec{k} \to -\vec{k} \Longrightarrow$

$G_F(x-x') = i\,\Theta(t'-t)\,\frac{1}{(2\pi)^3}\int\frac{e^{-ik\cdot x}}{2\omega_k}\,d^3k + i\,\Theta(t'-t)\,\frac{1}{(2\pi)^3}\int\frac{e^{ik\cdot x}}{2\omega_k}\,d^3k \Longrightarrow$

$G_F(x-x') = i\,\Theta(t'-t)\int e^{-ik\cdot x}\,d\tilde{k} + i\,\Theta(t-t')\int e^{ik\cdot x}\,d\tilde{k} \stackrel{(26)}{\Longrightarrow}$

$\boxed{G_F(x-x') = i\,\Theta(t'-t)\,\langle 0|\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle + i\,\Theta(t-t')\,\langle 0|\,\varphi^\dagger(x)\,\varphi(x')\,|0\rangle = i\langle 0|\hat{T}\,\varphi(x')\,\varphi^\dagger(x)\,|0\rangle}$ ... (28)

G_F is the amplitude describing the propagation of a particle from x' to x if $t > t'$, or from x to x' if $t' > t$.
Positive frequencies propagate forward in time, negative frequencies propagate backwards in time.

5.3 Quantization of the free Dirac Field

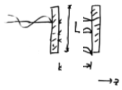
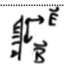
Lagrangian	$\boxed{\mathcal{L} = \bar\Psi(i\partial\!\!\!/ - m)\Psi}$ leads to $(i\partial\!\!\!/ - m)\Psi = 0$ and $\bar\Psi\left(i\overleftarrow{\partial\!\!\!/} + m\right) = 0$ when varied with respect to $\bar\Psi$ and Ψ
Conjugate momentum	$\pi_\Psi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Psi(x))} = \frac{\partial}{\partial(\partial_0\Psi(x))}\left(\bar\Psi(i\partial\!\!\!/ - m)\Psi\right) = \frac{\partial}{\partial(\partial_0\Psi(x))}\left(\bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi\right) = \frac{\partial}{\partial(\partial_0\Psi(x))}\left(i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi\right)$ $\pi_\Psi(x) = \frac{\partial}{\partial(\partial_0\Psi(x))}\left(i\bar\Psi(\gamma^0\partial_0\Psi + \gamma^i\partial_i\Psi) - m\bar\Psi\Psi\right) = i\bar\Psi\gamma^0 = i\Psi^\dagger\gamma^0\gamma^0 \Rightarrow \boxed{\pi_\Psi(x) = i\Psi^\dagger}$ ... (29)
Problem	We demand $[\Psi,\Psi^\dagger] \overset{!}{=} 0$ (i.e. Ψ and $\Psi^\dagger$ to be independent) and also $[\Psi,\pi_\Psi] \overset{(29)}{=} i[\Psi,\Psi^\dagger] \overset{!}{=} i\mathbb{1}$ ↯ contradiction!
Ansatz	$\boxed{\Psi_\alpha(x) = \int\sum_{a=1,2}\left(\hat b_a(k)\,u_\alpha^{(a)}(k)\,e^{-ik\cdot x} + \hat d_a^\dagger(k)\,v_\alpha^{(a)}(k)\,e^{ik\cdot x}\right)d\tilde k}$ ... (30a) with $d\tilde k \overset{\text{def}}{=} \frac{1}{(2\pi)^3}\frac{m}{\omega_k}dk$ and $k_0 = \omega_k$ $\boxed{\bar\Psi_\alpha(x) = \int\sum_{a=1,2}\left(\hat b_a^\dagger(k)\,\bar u_\alpha^{(a)}(k)\,e^{+ik\cdot x} + \hat d_a(k)\,\bar v_\alpha^{(a)}(k)\,e^{ik\cdot x}\right)d\tilde k}$ ... (30b) with $d\tilde k \overset{\text{def}}{=} \frac{1}{(2\pi)^3}\frac{m}{\omega_k}dk$ and $k_0 = \omega_k$
anti-commutators for fermions	$\boxed{[\hat b_a(k),\hat b_b^\dagger(k')]_+ = (2\pi)^3\frac{k^0}{m}\delta^3(\vec k - \vec k')\,\delta_{ab}}$ ... (31a) $\boxed{[\hat d_a(k),\hat d_b^\dagger(k')]_+ = (2\pi)^3\frac{k^0}{m}\delta^3(\vec k - \vec k')\,\delta_{ab}}$ ... (31b) all other = 0 $\boxed{[\Psi_\alpha(t,\vec x),\Psi_\beta(t,\vec y)]_+ = \delta^3(\vec x - \vec y)\,\delta_{\alpha\beta}}$ ... (31c) Consequence: take $a^\dagger(1)\,a^\dagger(2)\dots a^\dagger(n)\,\vert 0\rangle$ ... any swap $\Longrightarrow$ sign change (with 1…n standing for all possible indices like is it a b or d particle, is it spin up or down, momentum variable…)
Hamiltonian	$\hat H = \int(\pi\partial_0\Psi + \pi^\dagger\partial_0\Psi^\dagger - \mathcal{L})\,d^3x \Longrightarrow \cdots \Longrightarrow \boxed{:\hat H: = \int\omega_k\sum_{a=1,2}\left(\hat b_a^\dagger(k)\,\hat b_a(k) + \hat d_a^\dagger(k)\,\hat d_a(k)\right)d\tilde k}$ ... (32)
Spin statistics theorem	If we want to consistently quantize a relativistic field, we have to use Bose-Einstein statistics for integer spin fields and Fermi-Dirac statistics for half-integer spin fields. The Pauli exclusion principle is a consequence of the theory to be consistent.
anti-commut, unequal times	$\boxed{[\Psi_\alpha(x),\Psi_\beta(x')]_+ = (i\partial\!\!\!/_x + m)_{\alpha\beta}\,i\,\Delta(x-x')}$ with $i\,\Delta(x-x') \overset{\text{def}}{=} \int\left(\underbrace{e^{-ik\cdot(x-x')}}_{pos.energy} - \underbrace{e^{ik\cdot(x-x')}}_{neg.energy}\right)d\tilde k$ Again, causality is preserved by vanishing anti-commutators for space-like separations
Time-ordered product	$\hat T\,\Psi(x')\,\bar\Psi(x) \overset{\text{def}}{=} \Theta(t'-t)\,\Psi(x')\,\bar\Psi(x) - \Theta(t-t')\,\bar\Psi(x)\,\Psi(x')$ ... (33) ... different sign than scalar field
vacuum expectation value	$\boxed{\left\langle 0\left\vert\underbrace{\hat T\,\Psi_\alpha(x)\,\bar\Psi_\beta(y)}_{propagator}\right\vert 0\right\rangle \overset{\text{def}}{=} i\,S_F(x-y)_{\alpha\beta}}$ ... (34a) with $\boxed{S_F(x-y)_{\alpha\beta} = \frac{1}{(2\pi)^4}\int e^{-ik\cdot(x-y)}\underbrace{\frac{k\!\!\!/ + m}{k^2 - m^2 + i\varepsilon}}_{S_F(k) = \frac{i}{k\!\!\!/ - m + i\varepsilon}}d^4k}$ ... (34b)

5.4 Quantization of the free Electromagnetic Field

Lagrangian	$\boxed{\mathcal{L}(A,\partial A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$ ... (35) with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
Conjugate momentum	$\boxed{\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} = F^{\mu 0}}$ ... (36) $\pi^0 = F^{00} = 0$ ↯ quantization fails. Reason: Gauge freedom. The photon field has not 4 times as many degrees as a scalar field, but only twice as many.
Gauge breaking terms	$\boxed{\mathcal{L} \to \mathcal{L} - \frac{\lambda}{2}(\partial\cdot A)^2}$ ... (36) Result of $\mathcal{L}$ now depends on gauge choice. — Conjugate momentum: $\boxed{\pi^\mu = F^{\mu 0} - \lambda g^{\mu 0}(\partial\cdot A)}$ ... (37)
Commutator	Equal time: $[\hat A_\mu(t,\vec x),\pi^\nu(t,\vec y)] = i\delta_\mu^\nu\,\delta^3(\vec x - \vec y)$ ... (38) — Euler Lagrange: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0$ ... (39)
Field equtns	(36) in (38) $\Longrightarrow$ $\boxed{\Box\hat A_\mu - (1-\lambda)\partial_\mu\partial\cdot\hat A = 0}$ ... (40) — Feynman-gauge: $\lambda = 1 \overset{(39)}{\Longrightarrow} \Box\hat A_\mu = 0$
Expansion	$\boxed{A_\mu(x) = \int\sum_{\lambda=0}^{3}\left(a^{(\lambda)}(k)\,\varepsilon_\mu^{(\lambda)}(k)\,e^{-ik\cdot x} + a^{(\lambda)\dagger}(k)\,\varepsilon_\mu^{(\lambda)}(k)\,e^{ik\cdot x}\right)d\tilde k}$ ... (41) with $k_0 = \omega_k = \vert\vec k\vert$ and $\varepsilon_\mu^{(\lambda)}$ ... polarization vectors
Choice of polarization vectors	Assuming $k_\mu = (k,0,0,k)$: $\underbrace{\varepsilon_\mu^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{scalar\ polarization}$ $\underbrace{\varepsilon_\mu^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \varepsilon_\mu^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{transversal\ polarization}$ $\underbrace{\varepsilon_\mu^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{longitudinal\ polarization}$ — only transversal is physical Arbitrary direction of k_μ: $\sum_{\lambda=0}^{3}\frac{\varepsilon_\mu^{(\lambda)}(k)\,\varepsilon_\nu^{(\lambda)}(k)}{\varepsilon^{(\lambda)}(k)\cdot\varepsilon^{(\lambda)}(k)} = g_{\mu\nu}$ and $\varepsilon^{(\lambda)}(k)\cdot\varepsilon^{(\lambda')}(k) = g^{\lambda\lambda'}$ — Same-time commutator: $[\hat A_\mu(t,\vec x),\pi^\nu(t,\vec y)] = i\delta_\mu^\nu\,\delta^3(\vec x - \vec y)$
Conjugate momentum	$\pi^\mu = F^{\mu\nu} - \lambda g^{\mu 0}(\partial\cdot\hat A) = \underbrace{-\partial^0\hat A^\mu}_{time\ deriv.} + \underbrace{\partial^\mu\hat A^0 - g^{\mu 0}(\partial_\nu\hat A^\nu)}_{spatial\ derivatives} \overset{commutator}{\Longrightarrow} \left[\frac{\partial}{\partial t}\hat A_\mu(t,\vec x),\hat A_\nu(t,\vec y)\right] = i\delta_\mu^\nu\,\delta^3(\vec x - \vec y)$
Commutator	$\boxed{[a^{(\lambda)}(k),\hat a^{(\lambda')\dagger}(k')] = -g^{\lambda\lambda'}2\vert\vec k\vert(2\pi)^3\,\delta^3(\vec k - \vec k')}$ ... (42) only for $g^{\lambda\lambda'} = g^{00}$ we have a negative sign. $\boxed{[\hat A_\mu(x),\hat A_\nu(y)] = -ig_{\mu\nu}\,\Delta(x-y)}$ ... (43) ... vanishes for spacelike separations. Causality preserved.
Problem with Fock Space	With $\lambda = 0$: $\vert 1\rangle = \int f(k)\,\hat a^{(0)\dagger}(k)\,d\tilde k\,\vert 0\rangle = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_k}f(k)\,\hat a^{(0)\dagger}(k)\,d^3k \Longrightarrow$ $\langle 1\vert 1\rangle = \frac{1}{(2\pi)^6}\iint\frac{1}{2\omega_k}\frac{1}{2\omega_{k'}}f^*(k)\,f(k')\,\langle 0\vert\,\hat a^{(0)}(k)\,\hat a^{(0)\dagger}(k')\,\vert 0\rangle\,d^3k\,d^3k' \overset{(43)}{\Longrightarrow}$ $\langle 1\vert 1\rangle = \frac{1}{(2\pi)^3}\frac{1}{(2\pi)^3}\iint\frac{1}{2\omega_k}\frac{1}{2\omega_{k'}}f^*(k)\,f(k')\left(\langle 0\vert\,\hat a^{(0)\dagger}(k')\,\hat a^{(0)}(k)\,\vert 0\rangle - g^{00}2\omega_k(2\pi)^3\,\delta^3(\vec k - \vec k')\right)d^3k\,d^3k' \Longrightarrow$ $\langle 1\vert 1\rangle = -\int\vert f(k')\vert^2\,d\tilde k$ States generated by $a^{(0)\dagger}(k)$ can have a negative norm! ↯
Solution	Get rid of unphysical polarizations. Try $\partial\cdot A = \partial_\mu A^\mu = 0$ (Lorentz Gauge). But this is inconsistent with commutators. Try $\langle 0\vert\partial_\mu A^\mu\vert 0\rangle = 0$, but this is nonlinear. Actual solution to define a physical Hilbert space as a subspace of the unphysical Fock space is: $\boxed{\partial\cdot\hat A^{(+)}\vert\Psi\rangle = 0 \Longleftrightarrow \vert\Psi\rangle \in \mathcal{H}_1}$ ... (44) where $(+)$ means that only the annihilation operator part (the positive frequency part) of $\hat A$ is to be taken: $\boxed{\partial\cdot\hat A^{(+)} = -i\int e^{-ik\cdot x}\sum_{\lambda=0,3}\hat a^{(\lambda)}(k)\,\varepsilon^{(\lambda)}\cdot k\,d\tilde k}$ ... (45)

$\lvert\Psi\rangle \in \boldsymbol{\mathcal{H}}_1$	$\lvert\Psi\rangle = \lvert\Psi_T\rangle\lvert\emptyset\rangle$ with $\lvert\Psi_T\rangle$ generated by transverse creation operators, and $\lvert\emptyset\rangle$ by scalar and longitudinal ones. $\overset{(44)}{\Longrightarrow}$ $\partial\cdot A^{(+)}\lvert\Psi\rangle = \partial\cdot\hat{A}^{(+)}(\lvert\Psi_T\rangle\lvert\emptyset\rangle)\big\rvert\, \partial\cdot\hat{A}^{(+)}$ does not act on $\lvert\Psi_T\rangle \implies \boxed{[\hat{a}^{(0)}(k)-\hat{a}^{(3)}(k)]\lvert\emptyset\rangle = 0}$ ... (46) The states must have the same number of scalar fields as longitudinal fields. $\implies$ Let $\lvert\emptyset_n\rangle$ be a state with n scalar and longitudinal excitations. $\implies \hat{N}' = \int\left(\hat{a}^{(3)\dagger}(k)\,\hat{a}^{(3)}(k) - \hat{a}^{(0)\dagger}(k)\,\hat{a}^{(0)}(k)\right)d\tilde{k} \implies$ $\langle\emptyset_n\rvert\hat{N}'\lvert\emptyset_n\rangle = n\langle\emptyset_n\vert\emptyset_n\rangle = 0$
$:H:$	$:\hat{H}:=\int\left(\underbrace{\hat{a}^{(1)\dagger}(k)\,\hat{a}^{(1)}(k)+\hat{a}^{(2)\dagger}(k)\,\hat{a}^{(2)}(k)}_{\hat{H}_T}+\underbrace{\hat{a}^{(3)\dagger}(k)\,\hat{a}^{(3)}(k)-\hat{a}^{(0)\dagger}(k)\,\hat{a}^{(0)}(k)}_{\hat{H}_S}\right)\omega_k\,d\tilde{k}$ $\frac{\langle\Psi\rvert\hat{H}\lvert\Psi\rangle}{\langle\Psi\vert\Psi\rangle} = \frac{\langle\Psi\rvert\hat{H}_T\lvert\Psi\rangle}{\langle\Psi\vert\Psi\rangle} + \frac{\langle\Psi\rvert\hat{H}_S\lvert\Psi\rangle}{\langle\Psi\vert\Psi\rangle}\Big\rvert\, \lvert\Psi\rangle = \lvert\Psi_T\rangle\lvert\emptyset\rangle \implies \frac{\langle\Psi\rvert\hat{H}\lvert\Psi\rangle}{\langle\Psi\vert\Psi\rangle} = \frac{\langle\Psi_T\rvert\hat{H}_T\lvert\Psi_T\rangle\langle\emptyset\vert\emptyset\rangle}{\langle\Psi_T\vert\Psi_T\rangle\langle\emptyset\vert\emptyset\rangle} + \frac{\langle\Psi_T\vert\Psi_T\rangle\langle\emptyset\rvert\hat{H}_S\lvert\emptyset\rangle}{\langle\Psi_T\vert\Psi_T\rangle\langle\emptyset\vert\emptyset\rangle}\Big\rvert\, \langle\emptyset\rvert\hat{H}_S\lvert\emptyset\rangle = 0$ $\boxed{\frac{\langle\Psi\rvert\hat{H}\lvert\Psi\rangle}{\langle\Psi\vert\Psi\rangle} = \frac{\langle\Psi_T\rvert\hat{H}_T\lvert\Psi_T\rangle}{\langle\Psi_T\vert\Psi_T\rangle}}$ ... (46) only expectation value of physical states remain

5.4 Casimir Effect

Principle	Two conducting, large plates in vacuum are positioned parallel to each other in a small distance a. Between the two plates only certain wave modes are possible, whereas outside every mode is possible. The plates experience a net force pushing them together. 
wave factor	Between the plates: $k_z = \frac{n\pi}{a}$ with $n \neq 0$. k_x, k_y continuous. | For $k_z = 0$ only one mode possible 
Zero point energy	$E = \sum\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar c\sum\lvert\vec{k}\rvert = \frac{1}{2}\hbar c\sum\sqrt{\vec{k}_{\parallel}^2+k_z^2} = \frac{1}{2}\hbar c\sum\sqrt{\vec{k}_{\parallel}^2+\left(\frac{n\pi}{a}\right)^2}$ with $\vec{k}_{\parallel}^2 = k_x^2+k_y^2$ With boundary conditions:   $E = \frac{\hbar c}{2}\int\frac{L^2}{(2\pi)^2}\left(\underbrace{\lvert\vec{k}_{\parallel}\rvert}_{k_z=0} + \underbrace{2}_{polari-zations}\sum_{n=1}^{\infty}\sqrt{\vec{k}_{\parallel}^2+\left(\frac{n\pi}{a}\right)^2}\right)d^2k_{\parallel}$ ... divergent w/o boundary conditions: $E_0 = \frac{\hbar c}{2}\int\frac{L^2}{(2\pi)^2}\int\frac{a}{2\pi}2\sqrt{\vec{k}_{\parallel}^2+k_z^2}\,dk_z\,d^2k_{\parallel} = \frac{\hbar c}{2}\int\frac{L^2}{(2\pi)^2}\int 2\sqrt{\vec{k}_{\parallel}^2+\left(\frac{n\pi}{a}\right)^2}\,dn\,d^2k_{\parallel}$ ... also divergent
Regularization	Considering $E - E_0$, we assume that the boundary conditions of a perfect conductor apply only as long as the wavelength is larger than the typical atomic size R. We regulate the integrands with a cutoff function $f(\lvert\vec{k}\rvert)$. | $f(\lvert\vec{k}\rvert) = \begin{cases}1 \ldots \lvert\vec{k}\rvert \ll \frac{1}{R}\\ 0 \ldots \lvert\vec{k}\rvert \gg \frac{1}{R}\end{cases}$
Result	$\varepsilon \overset{\text{def}}{=} \frac{E-E_0}{L^2} = -\frac{\hbar c\pi^2}{720a^3}$ | Force per unit area | $\mathcal{F} = -\frac{\partial\varepsilon}{\partial a} = -\frac{\pi^2\hbar c}{240a^4} = -\underbrace{\frac{0.013}{(a[\mu m])^4}\,\text{dyn/cm}^2}_{0.1\text{Pa}}$ ... attractive force

6 Perturbation Theory

6.1 Interaction theory

Schrödinger picture	$\boxed{\lvert\Psi(t)\rangle_S = e^{-i\hat{H}(t-t_0)}\lvert\Psi(t_0)\rangle_S = e^{-i\hat{H}\bar{t}}\lvert\Psi(t_0)\rangle_S}$ ... (1) with $t - t_0 \overset{\text{def}}{=} \bar{t}$ fulfills Schrödinger eq. $\boxed{\frac{\partial}{\partial t}\lvert\Psi(t)\rangle_S = -i\hat{H}\lvert\Psi(t)\rangle_S}$ ... (2)
Heisenberg picture	$\lvert\Psi\rangle_H = \lvert\Psi(t_0)\rangle_S$ Time evolution is in the observables: $\boxed{\hat{O}_H(t) = e^{i\hat{H}\bar{t}}\hat{O}_S e^{-i\hat{H}\bar{t}}}$ ... (3) $\Longrightarrow \hat{O}_H(t_0) = \hat{O}_S$
Exp. value	${}_S\langle\chi(t)\lvert\hat{O}_S\rvert\Psi(t)\rangle_S = \langle\chi(t_0)\lvert\hat{O}_H(t)\rvert\Psi(t_0)\rangle$
Interaction picture	Separate time evolution of the free theory, and put all the other information is into the operators. $\hat{H} = \hat{H}_H = \hat{H}_S$ Separate: $\hat{H} = \hat{H}_{0S} + \hat{H}_{1S}$ with $\hat{H}_{0S}$ ... free part, $\hat{H}_{IS}$ ... interaction part $\hat{O}_I(t)$ ... operator of the free part, evolving as if there was no interaction $\boxed{\hat{O}_I(t) = e^{i\hat{H}_{0S}\bar{t}}\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}} = \hat{U}(t)\,\hat{O}_H\,\hat{U}^{-1}(t)}$ ... (4) with $\boxed{\hat{U}(t) = e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}}$ ... (5) ... time evolution in interaction picture = full time evolution "minus" evolution of free part Proof: $\hat{U}(t)\,\hat{O}_H\,\hat{U}^{-1}(t) \overset{(3)}{=} \hat{U}(t)\,e^{i\hat{H}\bar{t}}\hat{O}_S e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t) \overset{(5)}{=} e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}e^{i\hat{H}\bar{t}}\hat{O}_S e^{-i\hat{H}\bar{t}}e^{i\hat{H}\bar{t}}e^{-i\hat{H}_{0S}\bar{t}}$ $\boxed{\lvert\Psi(t)\rangle_I = e^{i\hat{H}_{0S}\bar{t}}\lvert\Psi(t)\rangle_S \overset{(3)}{=} e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}\lvert\Psi(t_0)\rangle_S \overset{(5)}{=} \hat{U}(t)\,\lvert\Psi(t_0)\rangle_S}$ ... (6)
Heisenberg equation	$\frac{\partial}{\partial t}\hat{O}_I(t) \overset{(4)}{=} \frac{\partial}{\partial t}\left(e^{i\hat{H}_{0S}\bar{t}}\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}}\right) = \left(\frac{\partial}{\partial t}e^{i\hat{H}_{0S}\bar{t}}\right)\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}} + e^{i\hat{H}_{0S}\bar{t}}\hat{O}_S\frac{\partial}{\partial t}e^{-i\hat{H}_{0S}\bar{t}} = i\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}} - ie^{i\hat{H}_{0S}\bar{t}}\hat{O}_S\hat{H}_{0S}e^{-i\hat{H}_{0S}\bar{t}} \Longrightarrow$ $\frac{\partial}{\partial t}\hat{O}_I(t) = i\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}} - ie^{i\hat{H}_{0S}\bar{t}}\hat{O}_S e^{-i\hat{H}_{0S}\bar{t}}\hat{H}_{0S} \overset{(4)}{=} i\hat{H}_{0S}\,\hat{O}_I(t) - i\,\hat{O}_I(t)\,\hat{H}_{0S} \Longrightarrow \boxed{\frac{\partial}{\partial t}\hat{O}_I(t) = i\left[\hat{H}_{0S}, \hat{O}_I(t)\right]}$ ... (7)
Schrödinger equation in the interaction picture	$i\frac{\partial}{\partial t}\lvert\Psi(t)\rangle_I \overset{\text{def}}{=} i\frac{\partial}{\partial t}\lvert t\rangle_I \overset{(6)}{=} i\frac{\partial}{\partial t}\left(e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S\right) = i\left(\frac{\partial}{\partial t}e^{i\hat{H}_{0S}\bar{t}}\right)\lvert t\rangle_S + e^{i\hat{H}_{0S}\bar{t}}i\frac{\partial}{\partial t}\lvert t\rangle_S = -\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S + e^{i\hat{H}_{0S}\bar{t}}i\frac{\partial}{\partial t}\lvert t\rangle_S \overset{(2)}{\Longrightarrow}$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = -\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S + e^{i\hat{H}_{0S}\bar{t}}i\left(-i\hat{H}\lvert t\rangle_S\right) = -\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S + e^{i\hat{H}_{0S}\bar{t}}\hat{H}\lvert t\rangle_S = -\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S + e^{i\hat{H}_{0S}\bar{t}}\left(\hat{H}_{0S} + \hat{H}_{1S}\right)\lvert t\rangle_S \overset{(6)}{\Longrightarrow}$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = -\hat{H}_{0S}\lvert t\rangle_I + e^{i\hat{H}_{0S}\bar{t}}\left(\hat{H}_{0S} + \hat{H}_{1S}\right)\lvert t\rangle_S = -\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}_{0S}\bar{t}}\lvert t\rangle_I + e^{i\hat{H}_{0S}\bar{t}}\left(\hat{H}_{0S} + \hat{H}_{1S}\right)e^{-i\hat{H}_{0S}\bar{t}}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S \Longrightarrow$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = -e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{0S}e^{-i\hat{H}_{0S}\bar{t}}\lvert t\rangle_I + e^{i\hat{H}_{0S}\bar{t}}\left(\hat{H}_{0S} + \hat{H}_{1S}\right)e^{-i\hat{H}_{0S}\bar{t}}e^{i\hat{H}_{0S}\bar{t}}\lvert t\rangle_S \overset{(6)}{\Longrightarrow}$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = -e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{0S}e^{-i\hat{H}_{0S}\bar{t}}\lvert t\rangle_I + e^{i\hat{H}_{0S}\bar{t}}\left(\hat{H}_{0S} + \hat{H}_{1S}\right)e^{-i\hat{H}_{0S}\bar{t}}\lvert t\rangle_I \Longrightarrow$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = \left(-e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{0S}e^{-i\hat{H}_{0S}\bar{t}} + e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{0S}e^{-i\hat{H}_{0S}\bar{t}} + e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{1S}e^{-i\hat{H}_{0S}\bar{t}}\right)\lvert t\rangle_I \Longrightarrow$ $i\frac{\partial}{\partial t}\lvert t\rangle_I = e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{1S}e^{-i\hat{H}_{0S}\bar{t}}\lvert t\rangle_I \Longrightarrow \boxed{i\frac{\partial}{\partial t}\lvert t\rangle_I = \hat{H}_{1I}(t)\,\lvert t\rangle_I}$ ... (8a) with $\boxed{\hat{H}_{1I}(t) = e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{1S}e^{-i\hat{H}_{0S}\bar{t}}}$ ... (8b)
$\hat{U}(t,t')$	$\boxed{\hat{U}(t-t') \overset{\text{def}}{=} \hat{U}(t,t') = \hat{U}(t)\,\hat{U}^{-1}(t')}$ ... (9a) $\Longrightarrow \boxed{\lvert t\rangle_I = \hat{U}(t,t')\,\lvert t'\rangle_I}$ ... (9b) $i\frac{d}{dt}\hat{U}(t,t') \overset{(9a)}{=} i\frac{d}{dt}\left(\hat{U}(t)\,\hat{U}^{-1}(t')\right) = i\left(\frac{d}{dt}\hat{U}(t)\right)\hat{U}^{-1}(t') \overset{(5)}{=} i\frac{d}{dt}\left(e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}\right)\hat{U}^{-1}(t') \Longrightarrow$ $i\frac{d}{dt}\hat{U}(t,t') = i\left(i\hat{H}_{0S}e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}} - ie^{i\hat{H}_{0S}\bar{t}}\hat{H}e^{-i\hat{H}\bar{t}}\right)\hat{U}^{-1}(t') = i\left(ie^{i\hat{H}_{0S}\bar{t}}\hat{H}_{0S}e^{-i\hat{H}\bar{t}} - ie^{i\hat{H}_{0S}\bar{t}}\hat{H}e^{-i\hat{H}\bar{t}}\right)\hat{U}^{-1}(t') \Longrightarrow$ $i\frac{d}{dt}\hat{U}(t,t') = ie^{i\hat{H}_{0S}\bar{t}}\left(i\hat{H}_{0S} - i\hat{H}\right)e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t')\,\Big\vert\, \hat{H} = \hat{H}_{0S} + \hat{H}_{1S} \Longrightarrow \frac{d}{dt}\hat{U}(t,t') = ie^{i\hat{H}_{0S}\bar{t}}\left(i\hat{H}_{0S} - i\hat{H}_{0S} - i\hat{H}_{1S}\right)e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t')$ $i\frac{d}{dt}\hat{U}(t,t') = e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{1S}e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t') = e^{i\hat{H}_{0S}\bar{t}}\hat{H}_{1S}e^{-i\hat{H}_{0S}\bar{t}}e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t') \overset{(8b)}{=} \hat{H}_{1I}(t)\,e^{i\hat{H}_{0S}\bar{t}}e^{-i\hat{H}\bar{t}}\,\hat{U}^{-1}(t') \overset{(5)}{\Longrightarrow}$ $i\frac{d}{dt}\hat{U}(t,t') = \hat{H}_{1I}(t)\,\hat{U}(t)\,\hat{U}^{-1}(t') \overset{(9a)}{\Longrightarrow} \boxed{i\frac{d}{dt}\hat{U}(t,t') = \hat{H}_{1I}(t)\,\hat{U}(t,t')}$ ... (10) (8a) $\Longrightarrow i\frac{d}{dt}\lvert t\rangle_I = \hat{H}_{1I}(t)\,\lvert t\rangle_I \Longrightarrow \frac{d}{dt}\lvert t\rangle_I = \frac{1}{i}\hat{H}_{1I}(t)\,\lvert t\rangle_I \Longrightarrow \int_{t'}^{t}\frac{d}{dt}\lvert t''\rangle_I\,dt'' = \frac{1}{i}\int_{t'}^{t}\hat{H}_{1I}(t'')\,\lvert t''\rangle_I\,dt''$ $\lvert t''\rangle_I\Big\vert_{t'}^{t} = \frac{1}{i}\int_{t'}^{t}\hat{H}_{1I}(t'')\,\lvert t'\rangle_I\,dt'' \Longrightarrow \lvert\lvert t\rangle_I - \lvert t'\rangle_I = \frac{1}{i}\int_{t'}^{t}\hat{H}_{1I}(t')\,\lvert t''\rangle_I\,dt'' \Longrightarrow \lvert t\rangle_I = \lvert t'\rangle_I + \frac{1}{i}\int_{t'}^{t}\hat{H}_{1I}(t'')\,\lvert t''\rangle_I\,dt'' \overset{(9b)}{\Longrightarrow}$ $\hat{U}(t,t')\,\lvert t'\rangle_I = \lvert t'\rangle_I + \frac{1}{i}\int_{t'}^{t}\hat{H}_{1I}(t'')\,\hat{U}(t'',t')\,dt''\,\lvert t'\rangle_I\,\Big\vert : \lvert t'\rangle_I \Longrightarrow \boxed{\hat{U}(t,t') = \mathbb{1} - i\int_{t'}^{t}\hat{H}_{1I}(t'')\,\hat{U}(t,t')\,dt''}$ ... (11) With $t_0 < t_3 < t_2 < t_1 < t$: $\hat{U}(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t}\hat{H}_{1I}(t_1)\,dt_1 + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_1}\hat{H}_{1I}(t_1)\,\hat{H}_{1I}(t_2)\,dt_2\,dt_1 + (-i)^3\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{t_0}^{t_2}\hat{H}_{1I}(t_1)\,\hat{H}_{1I}(t_2)\,\hat{H}_{1I}(t_3)\,dt_3\,dt_2\,dt_1$ ... $\hat{U}(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t}\hat{H}_{1I}(t_1)\,dt_1 + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t}\hat{T}\,\hat{H}_{1I}(t_1)\,\hat{H}_{1I}(t_2)\,dt_2\,dt_1 + \frac{(-i)^3}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\hat{T}\,\hat{H}_{1I}(t_1)\,\hat{H}_{1I}(t_2)\,\hat{H}_{1I}(t_3)\,dt_3\,dt_2\,dt_1$ ... $\hat{U}(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t}\hat{H}_{1I}(t')\,dt' + \frac{1}{2!}\left(-i\hat{T}\int_{t_0}^{t}\hat{H}_{1I}(t')\,dt'\right)^2 + \frac{1}{3!}\left(-i\hat{T}\int_{t_0}^{t}\hat{H}_{1I}(t')\,dt'\right)^3 + \cdots$ $\boxed{\hat{U}(t,t_0) = \hat{T}e^{-i\int_{t_0}^{t}\hat{H}_{1I}(t')\,dt'}}$ ... (12)

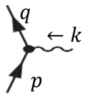
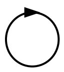
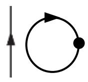
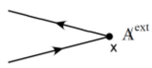
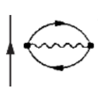
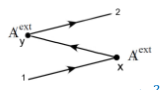
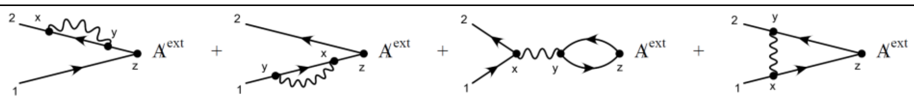
6.2 S-Matrix

Scattering	We have in-states and out-states, as coupling α is small, we assume these are well-separated at early and late times.
S-Matrix $S_{\beta\alpha}$ Scattering Operator $\hat{S}$	$\boxed{S_{\beta\alpha} = {}_H\langle\beta_{out}\lvert\alpha_{in}\rangle_H}$ ... (13) gives transition from very early to very late times We assume at $t \to -\infty$ we can "switch off" the interactions $\Longrightarrow \boxed{\lvert\alpha_{in}(-\infty)\rangle_I = \lvert\alpha_{in}\rangle_H}$ ... (14) $\Longrightarrow$ Incoming: $\boxed{\lvert\Psi(t)\rangle_I = \hat{U}(t,-\infty)\,\lvert\Psi\rangle_H}$ ... (15) At $t \to \infty$ we choose the basis for the out-states such that: $\boxed{\lvert\alpha_{out}(\infty)\rangle_I = \lvert\alpha_{in}(-\infty)\rangle_I = \lvert\alpha\rangle}$ ... (16) with orthogonality $\boxed{\langle\beta\vert\alpha\rangle = \delta_{\alpha\beta}}$ ... (17) Under these conditions: (13), (14) $\Longrightarrow S_{\beta\alpha} = {}_I\langle\beta_{out}(-\infty)\,\vert\,\alpha_{in}(-\infty)\rangle_I = {}_I\langle\beta_{out}(\infty)\,\vert\,\hat{U}(\infty,-\infty)\,\vert\,\alpha_{in}(-\infty)\rangle_I \Longrightarrow \boxed{S_{\beta\alpha} = \langle\beta\vert\hat{S}\vert\alpha\rangle}$ ... (18)
transtn matrix	If there is no interaction, then $\hat{S} = \mathbb{1} \Longrightarrow \boxed{S_{\beta\alpha} = \langle\beta\vert\hat{S}\vert\alpha\rangle = \delta_{\alpha\beta} + i\langle\beta\vert\hat{t}\vert\alpha\rangle}$ ... (19) with $\hat{t}$...transition matrix
In QED	$\boxed{\hat{S} = \hat{U}(\infty,-\infty) \overset{(12)}{=} \hat{T}e^{-i\int_{-\infty}^{\infty}\hat{H}_{1I}(t)\,dt}}$ ... (20) $\hat{H}_{1I} = \int\mathcal{H}_{1I}(t)\,d^3x = -\int\mathcal{L}_I(t)\,d^3x \overset{(20)}{\Longrightarrow} \boxed{\hat{S} = \hat{T}e^{i\int_{-\infty}^{\infty}\mathcal{L}_I(t)\,d^4x}}$ ... (21)

6.3 LSZ Reduction Technique

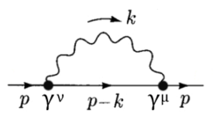
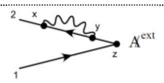
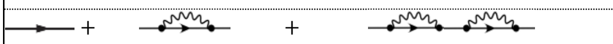
<table>
<tr><td>Assumption:</td><td>Neutral scalar field $\hat{b} = \hat{a}$ and $\hat{b}^\dagger = \hat{a}^\dagger$ │ Incoming states │ α_{in} given by $\alpha = \{\vec{p}_1, \dots, \vec{p}_n\}$ │ Outgoing state │ $\beta = \{\vec{q}_1, \dots, \vec{q}_n\}$
Pulling out one momentum $\vec{p} = \vec{p}_1$ from set α: $\alpha = \{\vec{p}, \alpha'\}$: (18) $\Longrightarrow S_{\beta\alpha} = \langle\beta|\hat{S}|\alpha\rangle = \langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p})\,|\alpha'\rangle \Longrightarrow$
$S_{\beta\alpha} = \langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p})\,|\alpha'\rangle + \langle\beta|\,\hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle - \langle\beta|\,\hat{a}^\dagger(\vec{p})\,\hat{S}|\alpha'\rangle \Longrightarrow \boxed{S_{\beta\alpha} = \langle\beta|\,\hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle + \langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle}$ ... $(22a)$
If $\vec{p} \notin \beta$: $\langle\beta|\,\hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle = 0$, else if $\vec{p} \in \beta$ ... $\langle\beta|\,\hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle$ is a disconnected contribution $\beta = \{\vec{p}, \beta'\}, \beta' = \beta\backslash\vec{p}$
In the following we can assume that α and β are disjoint.$\Longrightarrow \boxed{S_{\beta\alpha} = \langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle}$ ... $(22b)$</td></tr>
<tr><td>From 5.2.:</td><td>$\boxed{\varphi(t,\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_k}\left(\hat{a}(k)\,e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(k)\,e^{i\omega_k t - i\vec{k}\cdot\vec{x}}\right)d^3k}$ ... $(23a)$
$\boxed{\pi^\dagger(t,\vec{x}) = \frac{1}{(2\pi)^3}\int\left(\hat{a}(k)\,e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} - \hat{a}^\dagger(k)\,e^{i\omega_k t - i\vec{k}\cdot\vec{x}}\right)d^3k}$ ... $(23b)$</td></tr>
<tr><td>$\hat{a}^\dagger(k)$:</td><td>$\hat{a}^\dagger(k) = \int e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}(\omega_k\,\varphi(t,\vec{x}) - i\,\pi^\dagger(t,\vec{x}))\,d^3x$
$\hat{a}^\dagger(k) = \int\left(e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\omega_k\,\varphi(t,\vec{x}) - ie^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\,\pi^\dagger(t,\vec{x})\right)d^3x \,\Big|\, i\frac{\partial}{\partial t}e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} = \omega_k e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \Longrightarrow$
$\hat{a}^\dagger(k) = \int\left(\left(i\frac{\partial}{\partial t}e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\right)\varphi(t,\vec{x}) - ie^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\,\pi^\dagger(t,\vec{x})\right)d^3x \,\Big|\, \pi^\dagger(t,\vec{x}) = \frac{\partial}{\partial t}\varphi(t,\vec{x}) \Longrightarrow$
$\hat{a}^\dagger(k) = \int\left(\left(i\frac{\partial}{\partial t}e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\right)\varphi(t,\vec{x}) - ie^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\,\frac{\partial}{\partial t}\varphi(t,\vec{x})\right)d^3x = \int\left((i\partial_0 e^{-ip\cdot x})\,\varphi(x) - ie^{-ip\cdot x}\partial_0\,\varphi(x)\right)d^3x$
$\hat{a}^\dagger(k) = \int\left((i\partial_0 e^{-ip\cdot x}) - ie^{-ip\cdot x}\partial_0\right)\varphi(x)\,d^3x = \int\left(ie^{-ip\cdot x}\overleftarrow{\partial}_0 - ie^{-ip\cdot x}\overrightarrow{\partial}_0\right)\varphi(x)\,d^3x = \int e^{-ip\cdot x}\left(i\overleftarrow{\partial}_0 - i\overrightarrow{\partial}_0\right)\varphi(x)\,d^3x$
$\hat{a}^\dagger(k) = \int e^{-ip\cdot x}\left(-\frac{1}{i}\overleftarrow{\partial}_0 + \frac{1}{i}\overrightarrow{\partial}_0\right)\varphi(x)\,d^3x = \int e^{-ip\cdot x}\frac{1}{i}\left(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0\right)\varphi_I(x)\,d^3x \Longrightarrow$
$\boxed{\hat{a}^\dagger(k) = \int e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\,\varphi_I(x)\,d^3x \text{ with } \overleftrightarrow{\partial}_0 \stackrel{\text{def}}{=} \overrightarrow{\partial}_0 - \overleftarrow{\partial}_0}$ ... (24)</td></tr>
<tr><td>LSZ</td><td>Scattering between $t = -T$ and $t = t$. Let $-T' \ll -T$ and $T' \gg T$: $\stackrel{(22b)}{\Longrightarrow}$ $S_{\beta\alpha} = \langle\beta|\hat{S}\hat{a}^\dagger(\vec{p})|_{t=-T'} - \hat{a}^\dagger|_{t=T'}\hat{S}(\vec{p})\,|\alpha'\rangle$ ... (25)
$\langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle \stackrel{(24)}{=} \left(\int_{t=-T'}d^3x - \int_{t=T'}d^3x\right)e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\langle\beta|\hat{T}\,\varphi_I(x)\,\hat{S}|\alpha'\rangle$
$\langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle = -\int_{t=-T'}^{T'}d^4x\,\partial_0\left(e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\langle\beta|\hat{T}\,\varphi_I(x)\,\hat{S}|\alpha'\rangle\right)$ ... (26)
$\partial_0\left(e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\right)\varphi = \partial_0\left(e^{-ip\cdot x}\frac{1}{i}(\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)\varphi\right) = \partial_0\left(e^{-ip\cdot x}\frac{1}{i}(-\overleftarrow{\partial}_0 + \overrightarrow{\partial}_0)\varphi\right) = \frac{1}{i}\partial_0(-(\partial_0 e^{-ip\cdot x})\varphi + e^{-ip\cdot x}\partial_0\varphi) \Longrightarrow$
$\partial_0\left(e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\right)\varphi = \frac{1}{i}(-(\partial_0^2 e^{-ip\cdot x})\varphi - \cancel{(\partial_0 e^{-ip\cdot x})\partial_0\varphi + (\partial_0 e^{-ip\cdot x})\partial_0\varphi} + e^{-ip\cdot x}\partial_0^2\varphi) \Longrightarrow$
$\partial_0\left(e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\right) = \frac{1}{i}\left(e^{-ip\cdot x}\partial_0^2 - (\partial_0^2 e^{-ip\cdot x})\right) = \frac{1}{i}\left(e^{-ip\cdot x}\partial_0^2 - \left((-ip_0^2)e^{-ip\cdot x}\right)\right) = \frac{1}{i}\left(e^{-ip\cdot x}\partial_0^2 - \left((-E^2)e^{-ip\cdot x}\right)\right) \Longrightarrow$
$\partial_0\left(e^{-ip\cdot x}\frac{1}{i}\overleftrightarrow{\partial}_0\right) = \frac{1}{i}\left(e^{-ip\cdot x}\partial_0^2 - \left((-\vec{p}^2 - m^2)e^{-ip\cdot x}\right)\right) = \frac{1}{i}\left(e^{-ip\cdot x}\partial_0^2 - \left((\vec{\nabla}^2 - m^2)e^{-ip\cdot x}\right)\right) = \frac{1}{i}\left(e^{-ip\cdot x}(\Box + m^2)\right) \stackrel{(26)}{\Longrightarrow}$
$\boxed{\langle\beta|\hat{S}\,\hat{a}^\dagger(\vec{p}) - \hat{a}^\dagger\hat{S}(\vec{p})\,|\alpha'\rangle = i\int_{t=-T'}^{T'}e^{-ip\cdot x}(\Box + m^2)\,\langle\beta|\hat{T}\,\varphi_I(x)\,\hat{S}|\alpha'\rangle d^4x}$ ... (27) repeat for all momenta in the system $\Longrightarrow$
$\langle q_1 \dots q_m|\hat{S}|p_1 \dots p_n\rangle = i^{m+n}\int e^{q_1\cdot y_1}\dots e^{-p_n\cdot x_n}\left(\Box_{y_1} + m^2\right)\dots\left(\Box_{x_n} + m^2\right)\langle 0|\hat{T}\,\varphi_I(y_1)\dots\varphi_I(x_n)\,\hat{S}|0\rangle \stackrel{(21)}{\Longrightarrow}$
$\boxed{\langle q_1 \dots q_m|\hat{S}|p_1 \dots p_n\rangle = i^{m+n}\int e^{q_1\cdot y_1}\dots e^{-p_n\cdot x_n}\left(\Box_{y_1} + m^2\right)\dots\left(\Box_{x_n} + m^2\right)\left\langle 0\middle|\hat{T}\,\varphi_I(y_1)\dots\varphi_I(x_n)\,e^{i\int_{-\infty}^{\infty}\mathcal{L}_I(t)\,d^4x}\middle|0\right\rangle}$ ... (28)</td></tr>
</table>

6.4 Wick's Theorem and Feynman Rules

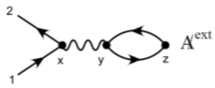
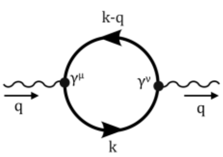
Wick's Theorem, contraction	$\hat{T}\varphi_1\varphi_2 = :\varphi_1\varphi_2: + \langle 0\vert\hat{T}\varphi_1\varphi_2\vert 0\rangle = :\varphi_1\varphi_2: - i\,G_F(x_1-x_2)$ ... (29) with $\varphi_i \overset{\text{def}}{=} \varphi(x_i)$ Generalization: $\hat{T}\varphi_1 \dots \varphi_n = :\varphi_1 \dots \varphi_n: + \sum_{k<l} :\varphi_1 \dots \hat{\varphi}_k \dots \hat{\varphi}_l \dots \varphi_n: \langle 0\vert\hat{T}\varphi_k\varphi_l\vert 0\rangle + \cdots$ $+ \sum_{k_1<k_2<\dots k_{2p}} :\varphi_1 \dots \hat{\varphi}_{k1} \dots \hat{\varphi}_{2p} \dots \varphi_n: \sum_P \langle 0\vert\hat{T}\varphi_{kP_1}\varphi_{kP_2}\vert 0\rangle \dots \langle 0\vert\hat{T}\varphi_{kP_{2P-1}}\varphi_{kP_{2P}}\vert 0\rangle + \cdots$ ... (30) where $\hat{\varphi}_k$ means omission of $\hat{\varphi}_k$, and Σ_P is the sum over all permutations. In words: A time-ordered product is the sum over all possible normal products where pairs of field operators are omitted and replaced by Feynman Green's function. This operation is called "contraction"
Example $\hat{T}\varphi_1\varphi_2\varphi_3\varphi_4$	$\hat{T}\varphi_1\varphi_2\varphi_3\varphi_4 =:\varphi_1\varphi_2\varphi_3\varphi_4: + :\varphi_3\varphi_4: \langle 0\vert\hat{T}\varphi_1\varphi_2\vert 0\rangle + :\varphi_2\varphi_4: \langle 0\vert\hat{T}\varphi_1\varphi_3\vert 0\rangle + :\varphi_2\varphi_3: \langle 0\vert\hat{T}\varphi_1\varphi_4\vert 0\rangle + :\varphi_1\varphi_4: \langle 0\vert\hat{T}\varphi_2\varphi_3\vert 0\rangle +$ $:\varphi_1\varphi_3: \langle 0\vert\hat{T}\varphi_2\varphi_4\vert 0\rangle + :\varphi_1\varphi_2: \langle 0\vert\hat{T}\varphi_3\varphi_4\vert 0\rangle + \langle 0\vert\hat{T}\varphi_1\varphi_2\vert 0\rangle\langle 0\vert\hat{T}\varphi_3\varphi_4\vert 0\rangle + \langle 0\vert\hat{T}\varphi_1\varphi_3\vert 0\rangle\langle 0\vert\hat{T}\varphi_2\varphi_4\vert 0\rangle +$ $\langle 0\vert\hat{T}\varphi_1\varphi_4\vert 0\rangle\langle 0\vert\hat{T}\varphi_2\varphi_3\vert 0\rangle$
LSZ application (scalar field)	Use in (30) in (28) for calculating $\left\langle 0\left\vert\hat{T}\,\varphi_I(y_1)\dots\varphi_I(x_n)\,e^{i\int_{-\infty}^{\infty}\mathcal{L}_I(t)\,d^4x}\right\vert 0\right\rangle$. One first expands $\hat{S} = e^{i\int_{-\infty}^{\infty}\mathcal{L}_I(t)\,d^4x}$ to some power in the coupling constants. It turns out that the building blocks $\langle 0\vert\hat{T}\varphi_1 \dots \varphi_{2P-1}\vert 0\rangle = 0$. What remains is: $\langle 0\vert\hat{T}\varphi_1 \dots \varphi_{2P}\vert 0\rangle = \Sigma_P\langle 0\vert\hat{T}\varphi_{kP_1}\varphi_{kP_2}\vert 0\rangle \dots \langle 0\vert\hat{T}\varphi_{kP_{2P-1}}\varphi_{kP_{2P}}\vert 0\rangle$ ... (31)
QED	We have the fermionic field Ψ, which becomes the electron, and the (vector potential of the) electromagnetic field A_μ, which becomes the photon.
Feynman propagators	electron: $\langle 0\vert\hat{T}\,\Psi_\alpha(x)\,\Psi_\beta(y)\,\vert 0\rangle = i\,S_F(x-y)_{\alpha\beta}$ $\Longrightarrow$ the fourier transformed version is denoted as —k→ photons: $\langle 0\vert\hat{T}\,A_\mu(x)\,A_\nu(y)\,\vert 0\rangle = -g_{\mu\nu}(-i\,G_F(x-y)) = -ig_{\mu\nu}\frac{1}{(2\pi)^4}\int\frac{1}{k^2+i\varepsilon}e^{-ik\cdot(x-y)}\,d^4k$ $\Longrightarrow$ denoted as ∿∿∿ γ
Interacting terms (Lagr.)	$\hat{S}$ built from $\mathcal{L}_I(x) = -e{:}\,\overline{\Psi}(x)\,\gamma^\mu\,A_\mu(x)\,\Psi(x){:} = -e{:}\,\overline{\Psi}(x)\,A\!\!\!/(x)\,\Psi(x){:}$ ... (32)  ... vertex Connects the incoming $\Psi(x)$ and the outgoing $\overline{\Psi}(x)$ fermion line with a photon line
Feynman rules	• Specified in momentum space • draw all possible (topological distinct) diagrams • for each vertex we get a factor of $-ie\gamma_\mu$ • for each vertex: momentum conservation e.g. $p+k=q$ • fermions propagator $k = \frac{i(p\!\!\!/-m)}{p^2-m^2+i\varepsilon} = \frac{i}{p\!\!\!/-m+i\varepsilon}$ • photon propagator: $-\frac{ig_{\mu\nu}}{p^2+i\varepsilon}$ • momenta through loops unbounded: $\frac{1}{(2\pi)^4}\int d^4k$ • closed fermion loop: additional factor -1 • symmetry factors  momentum conservation  closed fermion loop additional factor -1
vacuum diagrams	Contribute with an unobservable phase $\vert\langle 0\vert\hat{S}\vert 0\rangle\vert^2 = 1$ — order e^0:  — order e^2:  — order e^4: 
Electron propagating in a small external potential	Electron propagating an small external potential $A_\mu^{ext}(x)$ — In $\hat{S}$: $\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \underbrace{A_\mu^{ext}(x)\,\mathbb{1}}_{\substack{4-potential\\ not\ an\ operator}}$ — Leading **order e^0** contribution without interactions — $\langle 0\vert\hat{T}\Psi_1\Psi_2\vert 0\rangle$ excluded from LSZ  **order e^1**: 1 interaction — $-ie\left\langle 0\left\vert\hat{T}\overline{\Psi}_1\Psi_2\int_x :\overline{\Psi}_x\left(A\!\!\!/+A\!\!\!/^{ext}\right)_x\Psi_x:\right\vert 0\right\rangle$ (30) $\Longrightarrow A\!\!\!/$ cannot be paired, drops out — Combine $\overline{\Psi}_1\Psi_2$, $\overline{\Psi}_x\Psi_x$ no contribution  — Combine $\overline{\Psi}_1\Psi_x$, $\Psi_2\overline{\Psi}_x$  **order e^2**: two interactions $(-ie)^2\left\langle 0\left\vert\hat{T}\overline{\Psi}_1\Psi_2\int_x :\overline{\Psi}_x\left(A\!\!\!/+A\!\!\!/^{ext}\right)_x\Psi_x:\int_y :\overline{\Psi}_y\left(A\!\!\!/+A\!\!\!/^{ext}\right)_y\Psi_y:\right\vert 0\right\rangle$  excluded  also not relevant  small order $\left(A\!\!\!/^{ext}\right)^2$ **order e^3**: 3 interactions — $(-ie)^3\left\langle 0\left\vert\hat{T}\overline{\Psi}_1\Psi_2\int_x :\overline{\Psi}_x\left(A\!\!\!/+A\!\!\!/^{ext}\right)_x\Psi_x:\int_y :\overline{\Psi}_y\left(A\!\!\!/+A\!\!\!/^{ext}\right)_y\Psi_y:\int_z :\overline{\Psi}_z\left(A\!\!\!/+A\!\!\!/^{ext}\right)_z\Psi_z:\right\vert 0\right\rangle$ Relevant contributions:  fermion with electron self energy insertion at the outgoing line + fermion with electron self energy insertion at the incoming line + photon self energy insertion + vertex diagram
Complete QED Lagrangian	$\mathcal{L}_{QED} = \underbrace{\overline{\Psi}(i\partial\!\!\!/-m)\Psi}_{from\ Dirac\ field} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad -\tfrac{\lambda}{2}(\partial\cdot A)^2}_{from\ em\ field\ gauge\ breaking\ term} \overbrace{- \underbrace{e{:}\,\overline{\Psi}A\!\!\!/\Psi{:}}_{interaction} - \underbrace{e{:}\,\overline{\Psi}A\!\!\!/^{\text{ext}}\Psi{:}}_{\substack{external\ classical\\ 4-potential}}}^{\mathcal{L}_I}$ With gauge-covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ we can combine Dirac field and gauge breaking term: $\mathcal{L}_{QED} = \underbrace{\overline{\Psi}(iD\!\!\!/-m)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{gauge\ invariant} - e{:}\,\overline{\Psi}A\!\!\!/\Psi{:} - e{:}\,\overline{\Psi}A\!\!\!/^{\text{ext}}\Psi{:}$

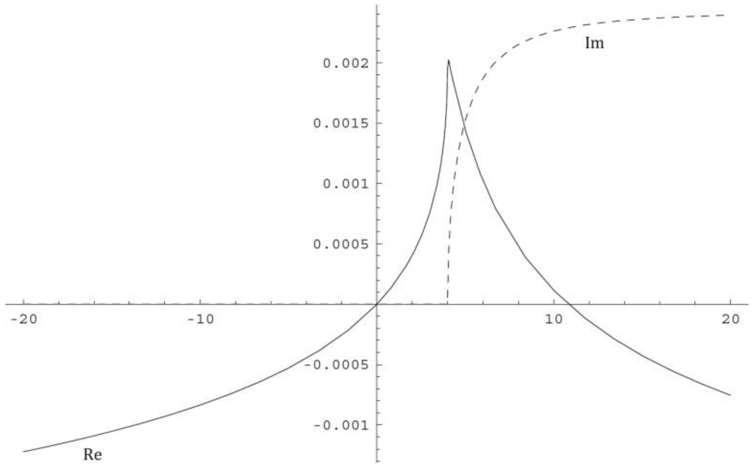
7 One-Loop Corrections

7.1 Fermion (e.g. Electron) Self Energy

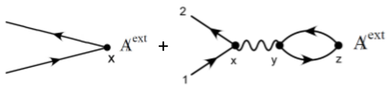
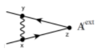
Electron self-energy insertions	 According to the Feynman rules: If we have such a loop with momentum k, then we have to integrate over all k. We cut off the external fermion lines: $\boxed{-i\Sigma(p) = (-ie)^2\frac{1}{(2\pi)^4}\int \underbrace{\gamma^\nu}_{vertex}\underbrace{\frac{i}{\not p-\not k-m+i\varepsilon}}_{fermion\ line}\underbrace{\gamma^\mu}_{vertex}\underbrace{\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}}_{photon\ propagator} d^4k}$ ... (33)
mass renormalization	Integration diverges linearly: $\Sigma \sim \int^\Lambda \frac{1}{k}\frac{1}{k^2}k^3\,dk \sim \int^\Lambda dk \sim \Lambda$
	Separate off the divergent part by expanding on the mass-shell $\not p^2 = m^2$ of the physical electron: $\Sigma(p) = \Sigma(p)\vert_{p=m} + \Sigma'(p)\vert_{p=m}(\not p-m) + \dots = \underbrace{A\mathbb{1}_4}_{linearly\ divergent} + \underbrace{B(\not p-m)}_{logarithmically\ divergent} + \underbrace{C(\not p-m)^2}_{convergent} + \cdots$ (34)  Incoming/outgoing fermion lines are on-shell
	 $+ \cdots \Longrightarrow$ $\frac{i}{\not p-m\varepsilon} + \frac{i}{\not p-m}(-i\Sigma(p))\frac{i}{\not p-m} + \frac{i}{\not p-m}(-i\Sigma(p))\frac{i}{\not p-m}(-i\Sigma(p))\frac{i}{\not p-m} + \cdots = \frac{i}{\not p-m}\sum_{n=0}^{\infty}\left(-i\Sigma(p)\frac{i}{\not p-m}\right)^n = \frac{i}{\not p-m}\frac{1}{1-\frac{-i\Sigma(p)i}{\not p-m}} =$ $\frac{i}{\not p-m}\frac{\not p-m}{\not p-m-\Sigma(p)} = \frac{i}{\not p-m-\Sigma(p)} \overset{(34)}{=} \frac{i}{\not p-m-A-B(\not p-m)+O((p-m)^2)} = \frac{i}{(1-B)(\not p-m)-A+O((p-m)^2)} = \frac{i\overbrace{\frac{1}{1-B}}^{1+B'}}{\not p-m-\underbrace{\frac{A}{1-B}}_{A'}+O((p-m)^2)} = \frac{i(1+B')}{\underbrace{\not p-m-A'}_{-m_{ren}}+O((p-m)^2)}$ We want $\boxed{m_{ren} = m + A' \approx m + A}$ ... (35) to be the physical mass of an electron. Since A is a divergent constant, m, the bare mass, must be likewise divergent to give a finite mass m_{ren}.
wave function renormaliztion	We absorb the divergent constant $1 + B' \stackrel{\text{def}}{=} Z_2$ into a renormalization of the spinor fields $\boxed{\Psi_{ren} = \frac{\Psi}{\sqrt{Z_2}}}$ ... (36)

7.2 Vacuum Polarization

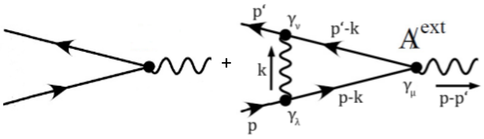
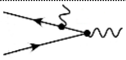
Introduction	 Before we had the scattering of an electron with a background em-field. In one of the diagrams (see left) there is an incoming electron, an outgoing electron, and the electron interacts at a vertex with a photon line which interacts with the creation of an electron/positron pair, which annihilate and interact with the external em-potential.
Vacuum polarization (photon self energy)	 $-\pi_{\mu\nu} = \underbrace{-}_{closed\ fermion\ loop}\underbrace{(-ie)^2}_{2\ vertices}\frac{1}{(2\pi)^4}\int \mathrm{tr}\overbrace{\left(\underbrace{\gamma_\mu}_{left\ vertex}\underbrace{\frac{i}{\not k-m+i\varepsilon}}_{lower\ fermion\ line}\underbrace{\gamma_\nu}_{right\ vertex}\underbrace{\frac{i}{\not k-\not q-m+i\varepsilon}}_{upper\ fermion\ line}\right)}^{1\ loop\ momentum\to integrate} d^4k$ Contrary to before where we required the fermion self-energy only on mass shell, since there is an intermediate photon line, we need the full function $\pi(q)$
Divergence	$\pi(q) \sim \int^\Lambda \underbrace{\frac{1}{k}\frac{1}{k}}_{fermion\ lines}\underbrace{k^3\,dk}_{d^4k\ integration} \sim \int^\Lambda k\,dk \sim \Lambda^2$ **Cut-Off regularization:** - analytic continuation to Euclidian momentum space $k_0 \to ik_4$ - cut-off by $-k^2 = \vec k^2 + k_4^2 < \Lambda^2$ - spoils gauge invariance in intermediate steps
Lattice cut-off regularization	- discretizing (Euclidean) spacetime with lattice - $\vert k_\mu\vert < \frac{\pi}{a}$ - difficult, only numerical, violates Lorentz invariance **Pauli-Villar regularizat.:** - introducing extra wrong-sign field with mass Λ^2 $\frac{1}{k^2-m^2} \to \frac{1}{k^2-m^2} - \frac{1}{k^2-\Lambda^2} = \frac{m^2-\Lambda^2}{(k^2-m^2)(k^2-\Lambda^2)} \sim \frac{1}{k^4}$ - Lorentz-covariant, but not well suited for photons
Dimensional regularization	- change number of spacetime dimensions from $n = 4$ to $\boxed{n = 4-\varepsilon}$ ... (37a) - we keep spinor dimensions to 4 - We have to use that $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = n$ ... (37b) $\pi_{\mu\nu} = (-ie)^2\frac{1}{(2\pi)^n}\int \mathrm{tr}\left(\gamma_\mu\frac{i}{\not k-m+i\varepsilon}\gamma_\nu\frac{i}{\not k-\not q-m+i\varepsilon}\right)d^nk = (-ie)^2\frac{1}{(2\pi)^n}\int\mathrm{tr}\left(\frac{-\gamma_\mu(\not k-\not q-m+i\varepsilon)\gamma_\nu(\not k-m+i\varepsilon)}{(\not k-m+i\varepsilon)(\not k-\not q-m+i\varepsilon)}\right)d^nk$ $\pi_{\mu\nu} = -ie^2\frac{1}{(2\pi)^n}\int\frac{\mathrm{tr}\left(\gamma_\mu(\not k+m\varepsilon)\gamma_\nu(\not k-\not q+m)\right)}{(\not k^2-m^2+i\varepsilon)((\not k-\not q)^2-m^2+i\varepsilon)}d^nk$ ... (38) $\Longrightarrow [m^2] = \left[e^2m^n\frac{m^2}{m^4}\right] \Longrightarrow [m^{4-n}] = [e^2] \Longrightarrow$ $\boxed{e^2 \to \tilde e^2 = e^2\mu^{4-n}}$ ... (39) with μ ... referential mass scale
Auxiliary calculation	$[\gamma_\mu,\gamma_\nu]_+ = 2g_{\mu\nu}\mathbb{1} \Longrightarrow \gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{1} \Longrightarrow \gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu+2g_{\mu\nu}\mathbb{1} \Longrightarrow \mathrm{tr}(\gamma_\mu\gamma_\nu) = -\mathrm{tr}(\gamma_\nu\gamma_\mu)+\mathrm{tr}(2g_{\mu\nu}\mathbb{1}) \Longrightarrow$ $\mathrm{tr}(\gamma_\mu\gamma_\nu) = -\mathrm{tr}(\gamma_\mu\gamma_\nu)+\mathrm{tr}(2g_{\mu\nu}\mathbb{1}) \Longrightarrow 2\,\mathrm{tr}(\gamma_\mu\gamma_\nu) = \mathrm{tr}(2g_{\mu\nu}\mathbb{1}) = 2g_{\mu\nu}\,\mathrm{tr}(\mathbb{1}) = 8g_{\mu\nu} \Longrightarrow \boxed{\mathrm{tr}(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu}}$ ... (40a) also: $\boxed{\mathrm{tr}(\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\rho) = 4(g_{\mu\sigma}g_{\mu\rho}+g_{\mu\rho}g_{\sigma\nu}-g_{\mu\nu}g_{\mu\sigma})} \overset{(38)}{\Longrightarrow} \pi_{\mu\nu} = -ie^2\frac{1}{(2\pi)^n}\int\frac{4(2k_\mu k_\nu-k_\mu q_\nu-k_\nu q_\mu+g_{\mu\nu}(m^2+k\cdot q-k^2))}{(\not k^2-m^2+i\varepsilon)((\not k-\not q)^2-m^2+i\varepsilon)}d^nk$ ... (41)
Feynman parametrization	$\frac{1}{ab} = \int_0^1\frac{1}{(ax+b(1-x))^2}$ ... (42) we define (see (41)) $a \stackrel{\text{def}}{=} ((\not k-\not q)^2-m^2+i\varepsilon), b \stackrel{\text{def}}{=} (\not k^2-m^2+i\varepsilon) \Longrightarrow$ $ax+b(1-x) = (k-xq)^2+q^2x(1-x)-m^2+i\varepsilon$ ... (43a) because we integrate from $k = -\infty$ to ∞, we can shift k : $k-xq \to \tilde k \Longrightarrow ax+b(1-x) = \tilde k^2+q^2x(1-x)-m^2+i\varepsilon$ ... (43b)
result	$\boxed{\pi_{\mu\nu}(q) = (g_{\mu\nu}q^2-q_\mu q_\nu)\frac{8\tilde e^2}{(4\pi)^{n/2}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 x(1-x)(m^2-q^2x(1-x))^{\frac{n}{2}-2} = (g_{\mu\nu}q^2-q_\mu q_\nu)\,\Pi(q^2)}$ ... (44)

Expansion	We expand terms from (44) at $n = 4 - \varepsilon$ for $\varepsilon \to 0$ • $\Gamma\left(2-\frac{n}{2}\right) = \Gamma\left(\frac{4-n}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma_E + O(\varepsilon)$ with $\frac{2}{\varepsilon}$ divergent, $\gamma_E = \lim_{n\to\infty}\left(-\ln(n) + \sum_{k=1}^{n}\frac{1}{k}\right)$ Euler–Mascheroni constant • $x^\alpha = \left(e^{\ln x}\right)^\alpha = e^{\alpha \ln x} = 1 + \alpha\ln(x) + \cdots \Longrightarrow x^{-\frac{\varepsilon}{2}} = 1 - \frac{\varepsilon}{2}\ln(x) + O(\varepsilon^2)$ • $\frac{1}{(4\pi)^{n/2}} = \frac{1}{(4\pi)^{2-\varepsilon/2}} = \frac{1}{(4\pi)^2}(4\pi)^{\frac{\varepsilon}{2}} = \frac{1}{16\pi^2}\left(1 + \frac{\varepsilon}{2}\ln(4\pi)\right) + O(\varepsilon^2)$ • $\tilde{e}^2 = e^2\mu^{4-n} = e^2\mu^\varepsilon = e^2\left(1 + \frac{\varepsilon}{2}\ln(\mu)\right) + \cdots$ $\overset{(44)}{\Longrightarrow} \ldots \Longrightarrow$ $\boxed{\Pi(q^2) = \frac{e^2}{2\pi^2}\left(\frac{1}{6}\left(\frac{2}{\varepsilon} - \gamma_E - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right) - \int_0^1 x(1-x)\ln\left(\frac{m^2 - q^2x(1-x)}{m^2}\right)dx\right) + O(\varepsilon)}$ … (45) … leading divergence: $\frac{2}{\varepsilon}$
Wavefunction renormaliz.	$\boxed{A^\mu = \sqrt{Z_3}A^\mu_{ren}}$ … (46a) $\Leftrightarrow$ $\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^{ren}_{\mu\nu}F^{\mu\nu}_{ren} - (Z_3 - 1)\frac{1}{4}F^{ren}_{\mu\nu}F^{\mu\nu}_{ren}}$ … (46b) $\boxed{Z_3 = 1 - \frac{e^2}{2\pi^2}\frac{1}{6}\left(\frac{2}{\varepsilon} + c\right)}$ … (46c)
Schemes	• Minimal subtraction scheme (MS): $c = 0$ … only the pole term of dimensional regularization is subtracted • Modified minimal subtraction ($\overline{\text{MS}}$): $c = -\gamma_E + \ln(4\pi)$ … removes the constants that invariably appear • Momentum subtraction (MOM): c is determined so that effective photon propagator = classical $\Pi^{MOM}(q^2 = -\mathcal{M}^2) = 0$ • On-shell (OS): Natural for QED. On the mass-shell of photons (the light cone q^2=0) there are no corrections $\Pi^{OS}(q^2 = 0)$=0
OS renor-malization	$q^2 = 0 \overset{(45)}{\Longrightarrow} \Pi^{OS}(q^2) = -\frac{e^2}{2\pi^2}\left(\int_0^1 x(1-x)\ln\left(\frac{m^2 - q^2x(1-x)}{m^2}\right)dx\right)$ … (47) $\Longrightarrow \cdots \Longrightarrow$ $\boxed{\Pi^{OS}(q^2) = \frac{e^2}{6\pi^2}\left(\frac{5}{6} + 2\frac{m^2}{q^2} - \left(1 + 2\frac{m^2}{q^2}\right)\sqrt{\frac{4m^2 - q^2}{q^2}}\arctan\sqrt{\frac{q^2}{4m^2 - q^2}}\right)}$ … (48) … for $q^2 < (2m)^2$ For $q^2 > (2m)^2$ there is also an imaginary part. It is related to the physical process of the decay of a virtual (off-shell) photon with $q^2 > (2m)^2$ into a real electron-positron pair. 

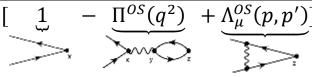
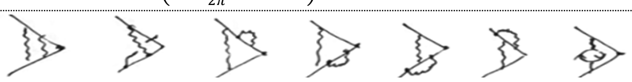
7.2.1 Uehling Potential

Considering	 An external potential receives a correction from the vacuum polarization diagram according to $\tilde{A}^{ext}_\mu(q) \to \left(\delta^\rho_\mu + \frac{-ig_{\mu\sigma}}{q^2}(-i)\,\Pi^{\sigma\rho}(q)\right)\tilde{A}^{ext}_\rho(q) \overset{(44)}{\Longrightarrow}$ $\tilde{A}^{ext}_\mu(q) \to \left(\delta^\rho_\mu - \delta^\rho_\mu\,\Pi(q^2) + \frac{q_\mu q_\rho}{q^2}\Pi(q^2)\right)\tilde{A}^{ext}_\rho(q) \Longrightarrow \boxed{\tilde{A}^{ext}_\mu(q) \to A^{ext}_\mu(q)\,(1-\Pi(q^2))}$ ... (48)
$\Pi^{OS}(-\vec{q}^{\,2})$	(47) $\Longrightarrow \Pi^{OS}(q^2) = -\frac{e^2}{2\pi^2}\left(\int_0^1 x(1-x)\ln\left(\frac{m^2-q^2x(1-x)}{m^2}\right)dx\right)$ | static case: $q_0 = 0 \Longrightarrow q^2 = -\vec{q}^{\,2} \Longrightarrow$ $\Pi^{OS}(-\vec{q}^{\,2}) = -\frac{e^2}{2\pi^2}\left(\int_0^1 x(1-x)\ln\left(\frac{m^2+\vec{q}^{\,2}x(1-x)}{m^2}\right)dx\right)$ $\Pi^{OS}(-\vec{q}^{\,2}) = -\frac{e^2}{2\pi^2}\left(\int_0^1 x(1-x)\ln\left(1+\frac{\vec{q}^{\,2}x(1-x)}{m^2}\right)dx\right)$ | nonrelativistic: $\frac{q^2}{m^2} \to 0 \Longrightarrow \ln\left(1+\frac{\vec{q}^{\,2}x(1-x)}{m^2}\right) \approx \frac{\vec{q}^{\,2}}{m^2}x(1-x)$ $\Pi^{OS}(-\vec{q}^{\,2}) = -\frac{e^2}{2\pi^2}\left(\int_0^1 x(1-x)\frac{\vec{q}^{\,2}}{m^2}x(1-x)\,dx\right) \Longrightarrow \boxed{\Pi^{OS}(-\vec{q}^{\,2}) = -\frac{e^2}{60\pi^2}\frac{\vec{q}^{\,2}}{m^2}}$ ... (49)
Uehling potential, nonrelativistic Coulomb potential + correction	(48) $\Longrightarrow \boxed{\frac{1}{\vec{q}^{\,2}} \to \frac{1}{\vec{q}^{\,2}}(1-\Pi(q^2))} \overset{(49)}{\Longrightarrow} V(\vec{q}) = \frac{1}{\vec{q}^{\,2}} + \frac{e^2}{60\pi^2m^2}$ ...in coordinate space: $V(\vec{x}) = \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\,V(\vec{q})\,d^3q = \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\left(\frac{1}{\vec{q}^{\,2}} + \frac{e^2}{60\pi^2m^2}\right)d^3q = \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\frac{1}{\vec{q}^{\,2}}d^3q + \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\frac{e^2}{60\pi^2m^2}d^3q \Longrightarrow$ $V(\vec{x}) = \frac{1}{(2\pi)^3}\iint q^2 e^{i\vec{q}\cdot\vec{x}}\frac{1}{\vec{q}^{\,2}}dq\,d\Omega + \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\frac{e^2}{60\pi^2m^2}d^3q \Longrightarrow V(\vec{x}) = \frac{1}{(2\pi)^3}\iint q^2 e^{i\lvert\vec{q}\rvert\lvert\vec{x}\rvert\cos\vartheta}\frac{1}{\vec{q}^{\,2}}dq\,d\Omega + \frac{1}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{x}}\frac{e^2}{60\pi^2m^2}d^3q$ $V(\vec{x}) = \frac{1}{(2\pi)^3}\iint q^2 e^{i\lvert\vec{q}\rvert\lvert\vec{x}\rvert\cos\vartheta}\frac{1}{\vec{q}^{\,2}}dq\,d\Omega + \frac{e^2}{60\pi^2m^2}\frac{1}{2\pi}\int e^{iq_xx}dq_x\frac{1}{2\pi}\int e^{iq_yy}dq_y\frac{1}{2\pi}\int e^{iq_zz}dq_z \Longrightarrow$ $V(\vec{x}) = \frac{1}{(2\pi)^3}\iint q^2 e^{i\lvert\vec{q}\rvert\lvert\vec{x}\rvert\cos\vartheta}\frac{1}{\vec{q}^{\,2}}dq\,d\Omega + \frac{e^2}{60\pi^2m^2}\delta(x)\,\delta(y)\,\delta(z) = \frac{1}{(2\pi)^3}\iint q^2 e^{i\lvert\vec{q}\rvert\lvert\vec{x}\rvert\cos\vartheta}\frac{1}{\vec{q}^{\,2}}dq\,d\Omega + \frac{e^2}{60\pi^2m^2}\delta^3(\vec{x}) \Longrightarrow$ $V(\vec{x}) = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} q^2\sin(\vartheta)\,e^{i\lvert\vec{q}\rvert\lvert\vec{x}\rvert\cos\vartheta}\frac{1}{\vec{q}^{\,2}}dq\,d\vartheta\,d\varphi + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2}$ | $\int_0^{\pi}\sin(\vartheta)\,d\vartheta = \int_{-1}^{1}ds$ with $s = \cos(\vartheta) \Longrightarrow$ $V(\vec{x}) = \frac{1}{4\pi^2}\int_{-1}^{1}\int_0^{\infty} e^{iqxs}\,dq\,ds + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2} = \frac{1}{4\pi^2}\int_0^{\infty}\frac{e^{iqx}-e^{-iqx}}{iqx}dq + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2} = \frac{1}{4\pi^2}\int_0^{\infty}\frac{2\sin(qx)}{qx}dq + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2}$ $V(\vec{x}) = \frac{1}{4\pi^2}\frac{1}{x}\int_0^{\infty}\frac{2\sin(qx)}{q}dq + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2} = \frac{1}{4\pi^2}\frac{1}{x}2\frac{\pi}{2} + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2} \Longrightarrow \boxed{V(\vec{x}) = \frac{1}{4\pi}\frac{1}{\lvert\vec{x}\rvert} + \delta^3(\vec{x})\frac{e^2}{60\pi^2m^2}}$ ... (50)
	Only corrections very close to the center. E.g. in Hydrogen-like atoms only the s-states ($l = 0$) are affected, since $\lvert\Psi_{n,l}(0)\rvert^2 = \frac{Z^3m^3\alpha^3}{\pi n^3}\delta_{l,0}$ First order perturbation theory leads to $\delta E_{n,l} = -\frac{Z\alpha e^2}{15\pi m^2}\delta_{l,0}\lvert\Psi_{n,0}(0)\rvert^2 = -\frac{4Z^4\alpha^5 m}{15\pi n^3}\delta_{l,0} \approx -27\mathrm{MHz}$ But experimental: $\approx +1000\mathrm{MHz}$. Reason: We need to take into account all diagrams, at least the vertex diagram 

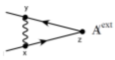
7.3 Vertex Diagram

Diagrams	 $\underbrace{-ie\,\Gamma_\mu(p',p)}_{\text{whole vertex}} = \underbrace{-ie\gamma_\mu}_{\text{leading term}}\ \underbrace{-ie\,\Lambda_\mu(p',p)}_{\text{loop correcton}}$ with
Divergence	$\underbrace{-ie\,\Lambda_\mu(p',p)}_{\text{loop correcton}} = \underbrace{(-i\tilde{e})^3}_{\text{3 vertices}}\overbrace{\frac{1}{(2\pi)^n}\int}^{\text{scaled coupling to dimensional regularization}}\underbrace{\frac{-ig^{\nu\lambda}}{k^2+i\varepsilon}}_{\text{photon propagator}}\underbrace{\gamma_\nu}_{\text{vertex}}\underbrace{\frac{i}{\not p'-\not k-m+i\varepsilon}}_{p'-k \text{ fermion line}}\underbrace{\gamma_\mu}_{\text{vertex}}\underbrace{\frac{i}{\not p-\not k-m+i\varepsilon}}_{p-k \text{ fermion line}}\underbrace{\gamma_\lambda}_{\text{vertex}}\,d^nk$ $\Lambda_\mu(p',p) = -(-i\tilde{e})^2 i^3\frac{1}{(2\pi)^n}\int\frac{g^{\nu\lambda}}{k^2+i\varepsilon}\gamma_\nu\frac{\not p'-\not k+m}{(\not p'-\not k)^2-m^2+i\varepsilon}\gamma_\mu\frac{\not p-\not k+m}{(\not p-\not k)^2-m^2+i\varepsilon}\gamma_\lambda\,d^nk$ $\Lambda_\mu(p',p) = (-i)(-i)(\tilde{e})^2 i\frac{1}{(2\pi)^n}\int\frac{g^{\nu\lambda}}{k^2+i\varepsilon}\gamma_\nu\frac{\not p'-\not k+m}{p'^2-2p'\cdot k+k^2-m^2+i\varepsilon}\gamma_\mu\frac{\not p-\not k+m}{p^2-2p\cdot k+k^2-m^2+i\varepsilon}\gamma_\lambda\,d^nk$ on-shell coulomb scattering $p^2=m^2=p'^2$ $\Lambda_\mu(p',p) = -i(\tilde{e})^2\frac{1}{(2\pi)^n}\int\frac{1}{k^2+i\varepsilon}\gamma^\lambda\frac{\not p'-\not k+m}{m^2-2p'\cdot k+k^2-m^2+i\varepsilon}\gamma_\mu\frac{\not p-\not k+m}{m^2-2p\cdot k+k^2-m^2+i\varepsilon}\gamma_\lambda\,d^nk$ $\lambda\to\nu$ $\bar{u}(p')\,\Lambda_\mu(p',p)\,u(p) = -i(\tilde{e})^2\frac{1}{(2\pi)^n}\int\frac{1}{k^2+i\varepsilon}\frac{1}{k^2-2k\cdot p'+i\varepsilon}\frac{1}{k^2-2k\cdot p+i\varepsilon}\bar{u}(p')\gamma^\nu\left(\not p'-\not k+m\right)\gamma_\mu(\not p-\not k+m)\gamma_\nu\,u(p)\,d^nk$ power counting: $\sim\int\frac{1}{k^2}\frac{1}{k}\frac{1}{k}k^3\,dk\sim\int\frac{1}{k}dk\sim\ln(k)$ divergent, even worse in dimensional regularization where $n=4-\varepsilon$ $\Longrightarrow$ introducing regulating mass term λ^2 $\bar{u}(p')\,\Lambda_\mu(p',p)\,u(p) = -i(\tilde{e})^2\frac{1}{(2\pi)^n}\int\frac{1}{k^2-\lambda^2+i\varepsilon}\frac{1}{k^2-2k\cdot p'+i\varepsilon}\frac{1}{k^2-2k\cdot p+i\varepsilon}\bar{u}(p')\gamma^\nu\left(\not p'-\not k+m\right)\gamma_\mu(\not p-\not k+m)\gamma_\nu\,u(p)\,d^nk$ ... (51) But, we have not just an UV divergence, but a new kind of divergence for low momenta – IR divergence!
Brems-strahlung	We also have  and . These photons can be emitted at arbitrarily soft momenta. IR divergences! Luckily, these IR divergences cancel exactly the IR divergences from above.
	• Because Λ_μ must be a Lorentz-vector and because at the same time it must be an object in spinor space, we can see that the general form must be: $\Lambda_\mu(p',p) = \gamma_\mu A + (p'_\mu+p_\mu)B + (p'_\mu-p_\mu)C$ ... (52) with A, B, C ... Lorentz scalar functions in spinor space • Because on-shell condition $p^2=m^2=p'^2$, the only non-trivial scalar is $q^2 = (p'_\mu-p_\mu)^2 = -2p'\cdot p+p'^2+p^2 \Longrightarrow$ $q^2 = -2p'\cdot p+2m^2$ ... (53) $\Longrightarrow A = \mathrm{A}(q^2), B = \mathrm{B}(q^2), C = \mathrm{C}(q^2)$ ... (54) • Because of gauge invariance Λ_μ and γ_μ satisfy $q^\mu\,\bar{u}(p')\,\Lambda_\mu(p',p)\,u(p) = 0$ ... (55) on mass shell • Whenever I have an expression $\bar{u}(p')\,(\not p-\not p)\,u(p) = m-m = 0 \overset{(52),(55)}{\Longrightarrow} q^\mu\,\bar{u}(p')\,\gamma_\mu A\,u(p) = 0$ • Similarly: $q^\mu(p'_\mu+p_\mu) = (p'-p)^\mu(p'+p)_\mu = 0 \Longrightarrow q^\mu\,\bar{u}(p')\,(p'_\mu+p_\mu)B\,u(p) = 0$ • Only $q^\mu\,\bar{u}(p')\,(p'_\mu-p_\mu)C\,u(p)\neq 0 \Longrightarrow$ in order for (55) to be fulfilled, we must set $C=0$ • Only two independent functions $\mathrm{A}(q^2)\equiv\mathrm{F}_1(q^2)$ and $B=\mathrm{B}(q^2)\equiv\mathrm{F}_2(q^2)$ ("structure functions") $\Longrightarrow$ $\Lambda_\mu(p',p) = \gamma_\mu\,\mathrm{F}_1(q^2)+(p'_\mu+p_\mu)\,\mathrm{F}_2(q^2)$ ... (56) • Using the Gordon identity $\bar{u}(p')\left(\not q(\not p-m)+\left(\not p'-m\right)\not q\right)u(p) = 0 \Longrightarrow \cdots \Longrightarrow$ $\bar{u}(p')\,(-2m\gamma_\mu+(p'_\mu+p_\mu)+i\sigma_{\mu\nu}q^\nu)\,u(p) = 0 \overset{(56)}{\Longrightarrow}$ • $\boxed{\Lambda_\mu(p',p) = \gamma_\mu\,\mathrm{F}_1(q^2)+\frac{i\sigma_{\mu\nu}q^\nu}{2m}\mathrm{F}_2(q^2)}$ ... (57)
renormalize	On-shell renormalization $\bar{u}(p')\,\Lambda_\mu^{OS}(p',p)\,u(p)\big\vert_{p^2=m^2=p'^2,\ p=p'} = 0 \Longrightarrow \boxed{\mathrm{F}_1^{ren}(0)=0}$ $\mathrm{F}_2(q^2)$ ... magnetic structure function
momentum transfer limit	If $q^2\ll m^2 \Longrightarrow \mathrm{F}_1^{ren}(q^2) = \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left(\ln\left(\frac{m}{\lambda}\right)-\frac{3}{8}\right)+\mathrm{O}\left(\frac{q^4}{m}\right)$ with $\alpha=\frac{e^2}{4\pi}$ $\mathrm{F}_2(q^2) = \frac{\alpha}{2\pi}\left(1+\frac{1}{6}\frac{q^2}{m^2}+\mathrm{O}\left(\frac{q^4}{m}\right)\right)$

7.4 Effective Interaction with a Weak External Field

Effective interaction	$-ie\gamma_\mu\,\tilde{A}^\mu_{ext}(q)\to -ie\{\gamma_\mu[\ \underbrace{1}\ -\ \underbrace{\Pi^{OS}(q^2)}\ +\underbrace{\Lambda_\mu^{OS}(p,p')}]\}\,\tilde{A}^\mu_{ext}(q)\Longrightarrow$  $-ie\gamma_\mu\,\tilde{A}^\mu_{ext}(q)\to -ie\left\{\gamma_\mu\left[1+\frac{\alpha q^2}{3\pi m^2}\left(\underbrace{\ln\left(\frac{m}{\lambda}\right)-\frac{3}{8}}_{F_1^{ren}}-\underbrace{\frac{1}{5}}_{\Pi^{OS}}\right)\right]+\underbrace{\frac{i}{2m}\frac{\alpha}{2\pi}}_{F_2^{ren}}\sigma_{\mu\nu}\,q^\nu\right\}\tilde{A}^\mu_{ext}(q)$
In configuration space	$q\to i\partial, q^\nu\to -i\partial^\nu, q^2\to -\Box_x, \tilde{A}^\mu_{ext}(q)\to\tilde{A}^\mu_{ext}(x)\ \Longrightarrow$ $-ie\gamma_\mu\,\tilde{A}^\mu_{ext}(x)\to -ie\left\{\gamma_\mu\left[1+\frac{\alpha q^2}{3\pi m^2}\left(\ln\left(\frac{m}{\lambda}\right)-\frac{3}{8}-\frac{1}{5}\right)\Box_x\right]-\frac{1}{2m}\frac{\alpha}{2\pi}\sigma_{\mu\nu}\partial^\nu\right\}\tilde{A}^\mu_{ext}(x)$
gyromag ratio	$g = 2(1+a) = 2\left(1+\frac{\alpha}{2\pi}+\mathrm{O}(\alpha^2)\right)\Longrightarrow a = 0.0011614\ldots$ (experiment: 0.00115965…)
$\mathrm{O}(\alpha^2)$	 ... 7 diagrams $\mathrm{O}(\alpha^3)$ ... 72 diagrams $\mathrm{O}(\alpha^4)$ ... 891 diagrams

7.4.2 Main Contribution to the Lamb Shift

	$-ie\gamma_\mu \tilde{A}^\mu_{ext}(x) \to -ie\left\{\gamma_\mu\left[1+\underbrace{\frac{\alpha q^2}{3\pi m^2}\left(\ln\left(\frac{m}{\lambda}\right)-\frac{3}{8}-\frac{1}{5}\right)\Box_x}_{Lamb\ shift\ contribution}\right]-\frac{1}{2m}\frac{\alpha}{2\pi}\sigma_{\mu\nu}\partial^\nu\right\}$
Estimation	Hydrogen Atom: From $E_{nj}=m(1-\frac{Z^2\alpha^2}{2n^2}-\frac{Z^4\alpha^4}{n^3(2j+1)}+\frac{3Z^4\alpha^4}{8n^4}+O(\alpha^6))$ we estimate $p^0=m\left(1-\frac{Z^2\alpha^2}{2n^2}\right)=m-V$ ... (1) Virial theorem: $\frac{\vec{p}^2}{2m}=\frac{1}{2}V \Longrightarrow \vec{p}^2=mV$ ... (2) Together: $\vec{p}^2-m^2=p_0^2-\vec{p}^2-m^2\overset{(1)}{=}(m-V)^2-\vec{p}^2-m^2\overset{(2)}{=}(m-V)^2-mV-m^2=m^2-2mV+V^2-mV-m^2 \Longrightarrow$ $\vec{p}^2-m^2=-3mV+V^2=-3mV+O(V^2)\Longrightarrow \boxed{\vec{p}^2-m^2=-3m^2\frac{Z^2\alpha^2}{2n^2}+O(\alpha^2)}$ ... (3)
	Before we assumed the electron being on-shell and had the propagator term $\frac{1}{k^2-2k\cdot p}$ coming from $\left.\frac{1}{k^2-2k\cdot p+p^2-m^2}\right\|_{p^2=m^2}$ But with (3) we get $\frac{1}{k^2-2k\cdot p+O(m^2\alpha^2)}$. The k integration is therefore not infrared divergent, but effectively cut off at photon momenta $k\sim m\alpha^2$ $\ln\left(\frac{m}{\lambda}\right)\to\ln\left(\frac{m}{m\alpha^2}\right)=\ln\left(\frac{1}{\alpha^2}\right)=\ln(137^2)\approx 10$ We arrive at the estimate $\boxed{\delta E_{n,l}\sim\frac{4Z^4\alpha^5 m}{3\pi n^3}\ln(137^2)\,\delta_{l,0}}$ for n=2 hydrogen atom Lamb shift $\approx$ 1300 MHz