Quantum Vacuum

(based on a lecture by Ulf Leonhardt and lecture notes by Lukas Rachbauer)
25.07.2022

1 Quantum Field Theory of Light

1.1 Light in Media

General	Light is a quantum field, described with Hermitian quantum Operators $\hat{F}(\vec{r},t)$. The media is described by dielectric functions $\varepsilon(\vec{r}),\mu(\vec{r})$. Simplifying assumptions: Media is non-dispersive and non-dissipative. We use the Heisenberg picture (state $\vert\psi\rangle$, or $\hat{\rho}$, respectively, is given), the evolution takes place in the observable $\hat{F}$: $\boxed{\partial_t\hat{F}=\frac{i}{\hbar}[\hat{H},\hat{F}]+\frac{\partial\hat{F}}{\partial t}}$ … (1) and $\frac{\partial\hat{\rho}}{\partial t}=0$ … (2)
Maxwell Classical with $\rho=0$	$\boxed{\vec{\nabla}\cdot\vec{B}=0}$ … (3a), $\boxed{\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}}$ … (3b), $\boxed{\vec{\nabla}\cdot\vec{D}=0}$ … (3c), $\boxed{\vec{\nabla}\times\vec{H}=\partial_t\vec{D}}$ … (3d) & boundary cond. & constitutive equations: $\boxed{\vec{D}=\varepsilon_0\varepsilon\vec{E}}$ … (4a), $\boxed{\vec{B}=\mu_0\mu\vec{H}}$ … (4b) Potential $\boxed{\vec{E}=-\partial_t\vec{A}}$ … (4c) $\boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$ … (4d)
Maxwell Quantum	Suppose $\vec{E}=\langle\hat{\vec{E}}\rangle,\vec{B}=\langle\hat{\vec{B}}\rangle\Longrightarrow\boxed{\vec{\nabla}\cdot\hat{\vec{B}}=0}$ … (5a), $\boxed{\vec{\nabla}\times\hat{\vec{E}}=-\partial_t\hat{\vec{B}}}$ … (5b), $\boxed{\vec{\nabla}\cdot\hat{\vec{D}}=0}$ … (5c), $\boxed{\vec{\nabla}\times\hat{\vec{H}}=\partial_t\hat{\vec{D}}}$ … (5d) Non-dispersive: $\boxed{\hat{\vec{D}}=\varepsilon_0\varepsilon\hat{\vec{E}}}$ … (6a), $\boxed{\hat{\vec{B}}=\mu_0\mu\hat{\vec{H}}}$ … (6b) with $\boxed{c^2=\frac{1}{\mu_0\varepsilon_0}}$ … (6c) because Maxwell equations are linear. E.g.: (5b) $\Longrightarrow\forall\psi$: $\langle\vec{\nabla}\times\hat{\vec{E}}+\partial_t\hat{\vec{B}}\rangle=\left\langle\psi\middle\vert\vec{\nabla}\times\hat{\vec{E}}+\partial_t\hat{\vec{B}}\middle\vert\psi\right\rangle=0\Longrightarrow$ if $\forall\psi$: $\left\langle\psi\middle\vert\hat{\vec{F}}\middle\vert\psi\right\rangle=0\wedge\hat{\vec{F}}$ Hermitian $\Longrightarrow\hat{\vec{F}}=0$ (all eigenvalues 0)
Potentials and Coulomb gauge fulfill Maxwell 5a-5c. One equation remains.	$\boxed{\hat{\vec{E}}=-\partial_t\hat{\vec{A}}}$ … (7a), $\boxed{\hat{\vec{B}}=\vec{\nabla}\times\hat{\vec{A}}}$ … (7b) automatically fulfills (5a): $\vec{\nabla}\cdot\hat{\vec{B}}\overset{(7b)}{=}\vec{\nabla}\cdot\left(\vec{\nabla}\times\hat{\vec{A}}\right)=0$ ✓ also automatically fulfills (5b): $\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}\overset{(7ab)}{\Longrightarrow}-\vec{\nabla}\times\left(\partial_t\hat{\vec{A}}\right)=-\partial_t\left(\vec{\nabla}\times\hat{\vec{A}}\right)\Longrightarrow-\partial_t\left(\vec{\nabla}\times\hat{\vec{A}}\right)=-\partial_t\left(\vec{\nabla}\times\hat{\vec{A}}\right)$ ✓ Coulomb gauge fix $\boxed{\vec{\nabla}\cdot\left(\varepsilon\hat{\vec{A}}\right)\overset{\text{def}}{=}0}$ … (8) fulfills (5c): $\vec{\nabla}\cdot\hat{\vec{D}}\overset{(6a)}{=}\varepsilon_0\vec{\nabla}\cdot\left(\varepsilon\hat{\vec{E}}\right)\overset{(7a)}{=}-\varepsilon_0\vec{\nabla}\cdot\left(\varepsilon\partial_t\hat{\vec{A}}\right)=-\varepsilon_0\partial_t\vec{\nabla}\cdot\left(\varepsilon\hat{\vec{A}}\right)\overset{(8)}{=}0$ ✓ Only remaining Maxwell equation (5d): $\vec{\nabla}\times\hat{\vec{H}}=\partial_t\hat{\vec{D}}\Longrightarrow\vec{\nabla}\times\hat{\vec{H}}-\partial_t\hat{\vec{D}}=0\overset{(6ab)}{\Longrightarrow}\frac{1}{\mu_0\mu}\vec{\nabla}\times\hat{\vec{B}}-\varepsilon_0\varepsilon\partial_t\hat{\vec{E}}=0\overset{(7ab)}{\Longrightarrow}$ $\frac{1}{\mu_0\mu}\vec{\nabla}\times\vec{\nabla}\times\hat{\vec{A}}+\varepsilon_0\varepsilon\partial_t^2\hat{\vec{A}}=0\Big\vert\cdot\frac{\mu_0}{\varepsilon}\Longrightarrow\frac{1}{\varepsilon\mu}\vec{\nabla}\times\vec{\nabla}\times\hat{\vec{A}}+\varepsilon_0\mu_0\partial_t^2\hat{\vec{A}}=0\overset{(6c)}{\Longrightarrow}\boxed{\frac{1}{\varepsilon}\vec{\nabla}\times\frac{1}{\mu}\vec{\nabla}\times\hat{\vec{A}}+\frac{1}{c^2}\partial_t^2\hat{\vec{A}}=0}$ … (9)
Hamiltonian	Heisenberg equation (1): $\partial_t\hat{F}=\frac{i}{\hbar}[\hat{H},\hat{F}]$ Hamiltonian: $\boxed{\hat{H}=\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{B}}\cdot\hat{\vec{H}}\right)dV}$ … (10a) $\overset{(6ab)}{\Longrightarrow}$ $\hat{H}=\frac{1}{2}\int\left(\frac{1}{\varepsilon_0\varepsilon}\hat{\vec{D}}^2+\frac{1}{\mu_0\mu}\hat{\vec{B}}^2\right)dV\overset{(7b)}{=}\frac{1}{2}\int\left(\frac{1}{\varepsilon_0\varepsilon}\hat{\vec{D}}^2+\frac{1}{\mu_0\mu}\frac{\varepsilon_0}{\varepsilon_0}\left(\vec{\nabla}\times\hat{\vec{A}}\right)^2\right)dV\overset{(6c)}{\Longrightarrow}\boxed{\hat{H}=\int\left(\frac{\hat{\vec{D}}^2}{2\varepsilon_0\varepsilon}+\frac{\varepsilon_0c^2}{2\mu}\left(\vec{\nabla}\times\hat{\vec{A}}\right)^2\right)dV}$ … (10b)
Commutators $[\hat{A}_l,\hat{D}_m]$ $\left[\hat{H},\hat{\vec{A}}(\vec{r}')\right]$ $\left[\hat{\vec{D}}(\vec{r}'),\hat{H}\right]$	Because of causality: $\left[\hat{\vec{\mathrm{A}}}(\vec{r}_1,t),\hat{\vec{\mathrm{A}}}(\vec{r}_2,t)\right]=0$ … (11a), $\left[\hat{\vec{\mathrm{D}}}(\vec{r}_1,t),\hat{\vec{\mathrm{D}}}(\vec{r}_2,t)\right]=0$ … (11b) Meaning: At a given time t, the fields $\hat{\vec{\mathrm{A}}}$ (or $\hat{\vec{\mathrm{D}}}$, respectively) at separated positions $\vec{r}_1$ and $\vec{r}_2$ are independent, and hence can be measured simultaneously.
	But: $\hat{\vec{\mathrm{A}}}$ and $\hat{\vec{\mathrm{D}}}$ are not independent: (6a) $\Longrightarrow\hat{\vec{D}}=\varepsilon_0\varepsilon\hat{\vec{E}}\overset{(7a)}{=}-\varepsilon_0\varepsilon\partial_t\hat{\vec{A}}\overset{(1)}{\Rightarrow}\hat{\vec{D}}=\frac{\varepsilon_0\varepsilon}{i\hbar}\left[\hat{H},\hat{\vec{A}}\right]$ … (12) $[\hat{A}_l,\hat{D}_m]\overset{(12)}{=}\frac{\varepsilon_0\varepsilon}{i\hbar}\left[\hat{A}_l,[\hat{H},\hat{A}_m]\right]=\frac{\varepsilon_0\varepsilon}{i\hbar}[\hat{A}_l,\hat{H}\hat{A}_m-\hat{A}_m\hat{H}]=\frac{\varepsilon_0\varepsilon}{i\hbar}\left(\hat{A}_l(\hat{H}\hat{A}_m-\hat{A}_m\hat{H})-(\hat{H}\hat{A}_m-\hat{A}_m\hat{H})\hat{A}_l\right)\Longrightarrow$ $[\hat{A}_l,\hat{D}_m]=\frac{\varepsilon_0\varepsilon}{i\hbar}\left(\hat{A}_l\hat{H}\hat{A}_m-\hat{A}_l\hat{A}_m\hat{H}-\hat{H}\hat{A}_m\hat{A}_l+\hat{A}_m\hat{H}\hat{A}_l\right)\Big\vert\;\hat{A}_l\hat{A}_m=\hat{A}_m\hat{A}_l$ (all spatial components independent) … (13) $\Longrightarrow$ $[\hat{A}_l,\hat{D}_m]=\frac{\varepsilon_0\varepsilon}{i\hbar}\left(\hat{A}_m\hat{H}\hat{A}_l-\hat{A}_m\hat{A}_l\hat{H}-\hat{H}\hat{A}_l\hat{A}_m+\hat{A}_l\hat{H}\hat{A}_m\right)\Big\vert$ compare with (13) $\Longrightarrow\boxed{[\hat{A}_l,\hat{D}_m]=[\hat{A}_m,\hat{D}_l]}$ … (14) …symmetric
	$[\hat{H},\hat{A}_m(\vec{r}')]\overset{(10b)}{=}\left[\int\left(\frac{\hat{\vec{D}}^2(\vec{r})}{2\varepsilon_0\varepsilon}+\frac{\varepsilon_0c^2}{2\mu}\left(\vec{\nabla}\times\hat{\vec{A}}\right)^2\right)d^3r,\hat{A}_m(\vec{r}')\right]=\frac{1}{2\varepsilon_0\varepsilon}\int\left[\hat{\vec{D}}^2(\vec{r}),\hat{A}_m(\vec{r}')\right]d^3r+\frac{1}{2\varepsilon_0\varepsilon}\int\underbrace{\left[\left(\vec{\nabla}\times\hat{\vec{A}}(\vec{r})\right)^2,\hat{A}_m(\vec{r}')\right]}_{=0\text{ because of }(11a)}d^3r$ $[\hat{H},\hat{A}_m(\vec{r}')]=\frac{1}{2\varepsilon_0\varepsilon}\int\left[\hat{\vec{D}}(\vec{r})\cdot\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]d^3r=\frac{1}{2\varepsilon_0\varepsilon}\int\left(\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]+\underbrace{\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]}_{\substack{=[D^\dagger,A^\dagger]=D^\dagger A^\dagger-A^\dagger D^\dagger\\=(AD-DA)^\dagger=-[D,A]^\dagger}}\cdot\underbrace{\hat{\vec{D}}(\vec{r})}_{=D^\dagger}\right)d^3r$ $[\hat{H},\hat{A}_m(\vec{r}')]=\frac{1}{2\varepsilon_0\varepsilon}\int\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]d^3r-\frac{1}{2\varepsilon_0\varepsilon}\int\left(\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]^\dagger\cdot\hat{\vec{D}}^\dagger(\vec{r})\right)d^3r\Longrightarrow$ $\boxed{[\hat{H},\hat{A}_m(\vec{r}')]=\frac{1}{2\varepsilon_0\varepsilon}\int\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]d^3r-\left(\frac{1}{2\varepsilon_0\varepsilon}\int\left(\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{A}_m(\vec{r}')\right]\right)d^3r\right)^\dagger}$ … (15a) in vector notation: $\boxed{\left[\hat{H},\hat{\vec{A}}(\vec{r}')\right]=\frac{1}{2\varepsilon_0\varepsilon}\int\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{\vec{A}}(\vec{r}')\right]d^3r-\left(\frac{1}{2\varepsilon_0\varepsilon}\int\left(\hat{\vec{D}}(\vec{r})\cdot\left[\hat{\vec{D}}(\vec{r}),\hat{\vec{A}}(\vec{r}')\right]\right)d^3r\right)^\dagger}$ … (15b)
	Analogously: $\boxed{\left[\hat{\vec{D}}(\vec{r}'),\hat{H}\right]=\frac{1}{2}\int\left(\left[\hat{\vec{D}}(\vec{r}'),\hat{\vec{A}}(\vec{r})\right]\cdot\vec{\nabla}\times\hat{H}(\vec{r})\right)d^3r-\left(\frac{1}{2}\int\left(\left[\hat{\vec{D}}(\vec{r}'),\hat{\vec{A}}(\vec{r})\right]\cdot\vec{\nabla}\times\hat{H}(\vec{r})\right)d^3r\right)^\dagger}$ … (16)
We expect	(5d) $\Longrightarrow\vec{\nabla}\times\hat{\vec{H}}=\partial_t\hat{\vec{D}}\overset{(1)}{=}\frac{i}{\hbar}\left[\hat{H},\hat{\vec{D}}\right]=\frac{1}{i\hbar}\left[\hat{\vec{D}},\hat{H}\right]\Longrightarrow\boxed{\left[\hat{\vec{D}},\hat{H}\right]=i\hbar\vec{\nabla}\times\hat{\vec{H}}}$ … (17a) (6a) $\Longrightarrow\hat{\vec{D}}=\varepsilon_0\varepsilon\hat{\vec{E}}\overset{(7a)}{=}-\varepsilon_0\varepsilon\partial_t\hat{\vec{A}}\overset{(1)}{=}-\varepsilon_0\varepsilon\frac{i}{\hbar}\left[\hat{H},\hat{\vec{A}}\right]=\frac{\varepsilon_0\varepsilon}{i\hbar}\left[\hat{H},\hat{\vec{A}}\right]\Longrightarrow\boxed{\left[\hat{H},\hat{\vec{A}}\right]=\frac{i\hbar}{\varepsilon_0\varepsilon}\hat{\vec{D}}}$ … (17b)

Commutator $\left[\hat{\vec D}(\vec r'),\hat{\vec A}(\vec r)\right]$	(16) in $(17a)\Longrightarrow\frac{1}{2}\int\left(\left[\hat{\vec D}(\vec r'),\hat{\vec A}(\vec r)\right]\cdot\vec\nabla\times\hat{\vec H}(\vec r)\right)d^3r-\left(\frac{1}{2}\int\left(\left[\hat{\vec D}(\vec r'),\hat{\vec A}(\vec r)\right]\cdot\vec\nabla\times\hat{\vec H}(\vec r)\right)d^3r\right)^\dagger=i\hbar\,\vec\nabla\times\hat{\vec H}$ Fulfilled by $\boxed{\left[\hat{\vec D}(\vec r',t),\hat{\vec A}(\vec r,t)\right]=i\hbar\,\delta^\perp(\vec r'-\vec r)}$ ... (18) Proof: $\frac{1}{2}\int\left(i\hbar\,\delta^\perp(\vec r'-\vec r)\cdot\vec\nabla\times\hat{\vec H}(\vec r)\right)d^3r$ - $\left(\frac{1}{2}\int\left(i\hbar\,\delta^\perp(\vec r'-\vec r)\cdot\vec\nabla\times\hat{\vec H}(\vec r)\right)d^3r\right)^\dagger=\frac{i\hbar}{2}\vec\nabla\times\hat{\vec H}(\vec r')-\left(\frac{i\hbar}{2}\vec\nabla\times\hat{\vec H}(\vec r')\right)^\dagger=i\hbar\vec\nabla\times\hat{\vec H}(\vec r')$ ✓ (16) in $(17b)\Longrightarrow\frac{1}{2\varepsilon_0\varepsilon}\int\hat{\vec D}(\vec r)\cdot\left[\hat{\vec D}(\vec r),\hat{\vec A}(\vec r')\right]d^3r-\left(\frac{1}{2\varepsilon_0\varepsilon}\int\left(\hat{\vec D}(\vec r)\cdot\left[\hat{\vec D}(\vec r),\hat{\vec A}(\vec r')\right]\right)d^3r\right)^\dagger=\frac{i\hbar}{\varepsilon_0\varepsilon}\hat{\vec D}$ ... also fulfilled by (18): $\frac{1}{2\varepsilon_0\varepsilon}\int\hat{\vec D}(\vec r)\cdot i\hbar\,\delta^\perp(\vec r'-\vec r)\,d^3r-\left(\frac{1}{2\varepsilon_0\varepsilon}\int\left(\hat{\vec D}(\vec r)\cdot i\hbar\,\delta^\perp(\vec r'-\vec r)\right)d^3r\right)^\dagger=\frac{i\hbar}{2\varepsilon_0\varepsilon}\hat{\vec D}(\vec r')+\left(\frac{i\hbar}{2\varepsilon_0\varepsilon}\hat{\vec D}(\vec r')\right)^\dagger=\frac{i\hbar}{\varepsilon_0\varepsilon}\hat{\vec D}(\vec r')$ ✓
Transverse Delta Function $\delta^\perp(\vec r-\vec r')$	$\delta^\perp$ is called the “Transverse Delta Function”. It is not an ordinary matrix delta-function: $\delta^\perp(\vec r'-\vec r)\neq\delta(\vec r'-\vec r)\,\mathbb{1}$. Properties: (a) $\delta^\perp$ must act like an ordinary delta function on $\hat{\vec D}$ and $\vec\nabla\times\hat{\vec H}$ on divergence-free (“transverse”) vector fields. (b) Because of equations (14) and (18), $\delta^\perp$ must be symmetric: $\boxed{\delta^\perp_{lm}=\delta^\perp_{ml}}$ ... $(19a)$ (c) Consider the divergence of equation (18) with regards to $\vec r'$: $\vec\nabla_{,}\cdot\left[\hat{\vec D}(\vec r'),\hat{\vec A}(\vec r)\right]=i\hbar\vec\nabla'\cdot\delta^\perp(\vec r'-\vec r)\Longrightarrow$ $\vec\nabla'\cdot\left(\hat{\vec D}(\vec r')\,\hat{\vec A}(\vec r)\right)-\vec\nabla'\cdot\left(\hat{\vec A}(\vec r)\,\hat{\vec D}(\vec r')\right)=i\hbar\vec\nabla'\cdot\delta^\perp(\vec r'-\vec r)\overset{(5c)}{\Longrightarrow}0=\vec\nabla'\cdot\delta^\perp(\vec r'-\vec r)\Longrightarrow\boxed{\vec\nabla\cdot\delta^\perp(\vec r)=0}$ ... $(19b)$
	Consider $\vec\nabla\times\hat{\vec H}(\vec r)=\int_{\mathbb{R}^3}\underbrace{\delta^\perp(\vec r-\vec r')}_{v}\underbrace{\vec\nabla'\times\hat{\vec H}(\vec r')}_{u'}d^3r'=uv\big\vert_{-\infty}^{\infty}-\int_{\mathbb{R}^3}uv'\,d^3r'\Big\vert$ with $u\overset{\text{def}}{=}\hat{\vec H}(\vec r'),\ v'\overset{\text{def}}{=}\vec\nabla_{,}\times\delta^\perp(\vec r-\vec r')\Longrightarrow$ $\vec\nabla\times\hat{\vec H}=\cancel{\hat{\vec H}(\vec r')\,\delta^\perp(\vec r-\vec r')}\Big\vert_{-\infty}^{\infty}-\int_{\mathbb{R}^3}\hat{\vec H}(\vec r')\,\vec\nabla'\times\delta^\perp(\vec r-\vec r')\,d^3r'=\int_{\mathbb{R}^3}\hat{\vec H}(\vec r')\,\vec\nabla\times\delta^\perp(\vec r-\vec r')\,d^3r'$ But this also works with the ordinary matrix delta function $\delta\mathbb{1}$ instad of $\delta^\perp$. Therefore: $\vec\nabla\times\delta^\perp=\vec\nabla\times(\delta\mathbb{1})\Longrightarrow\vec\nabla\times(\delta^\perp-\delta\mathbb{1})=0\Longrightarrow\vec\nabla\times\left(\delta^\perp-\delta\mathbb{1}-\vec\nabla\cdot h\right)=0\Longrightarrow\vec\nabla\times\delta^\perp=\vec\nabla\times\left(\delta\mathbb{1}+\vec\nabla\cdot h\right)\Longrightarrow$ $\delta^\perp(\vec r-\vec r')=\delta(\vec r-\vec r')\,\mathbb{1}+\vec\nabla\cdot h(\vec r-\vec r')$ ... $(20a)$ In index notation: $\delta^\perp_{lm}(\vec r-\vec r')=\delta_{lm}(\vec r-\vec r')+\partial_l\,h_m(\vec r-\vec r')$ ... $(20b)$
	Fourier Transform with $k=\left\lvert\vec k\right\rvert$ and some function $k_lk_m\,f(k)$ (k_lk_m because of symmetry equation $(19)\Longrightarrow\delta^\perp_{lm}=\delta^\perp_{ml}$) $\delta^\perp_{lm}(\vec r)=\frac{1}{(2\pi)^3}\int e^{i\vec k\cdot\vec r}\left(\delta_{lm}-k_lk_m\,f(k)\right)d^3k$ ... (21) $(19)\Longrightarrow\left(\vec\nabla\cdot\delta^\perp(\vec r)\right)_m=0\Longrightarrow\partial_l\,\delta^\perp_{lm}(\vec r)=0\overset{(21)}{\Longrightarrow}$ $\frac{1}{(2\pi)^3}\partial_l\int e^{i\vec k\cdot\vec r}\left(\delta_{lm}-k_lk_m\,f(k)\right)d^3k=\frac{1}{(2\pi)^3}\int ik_le^{i\vec k\cdot\vec r}\left(\delta_{lm}-k_lk_m\,f(k)\right)d^3k=\frac{1}{(2\pi)^3}\int ie^{i\vec k\cdot\vec r}\left(k_l\delta_{lm}-k_lk_lk_m\,f(k)\right)d^3k=0$ $\Longrightarrow\frac{1}{(2\pi)^3}\int ie^{i\vec k\cdot\vec r}\left(k_m-k^2k_m\,f(k)\right)d^3k=0\Longrightarrow k_m-k^2k_m\,f(k)=0\Longrightarrow f(k)=\frac{1}{k^2}\overset{(21)}{\Longrightarrow}$ $\boxed{\delta^\perp_{lm}(\vec r)=\frac{1}{(2\pi)^3}\int e^{i\vec k\cdot\vec r}\left(\delta_{lm}-\frac{k_lk_m}{k^2}\right)d^3k}$ ... (22)

1.2 Light Modes

General	In classical physics, if $\vec A_1(\vec r,t)$ and $\vec A_2(\vec r,t)$ are solutions to Maxwell’s equations, then also any superposition $\alpha_1\vec A_1(\vec r,t)+\alpha_2\vec A_2(\vec r,t)$ is a solution. However, in quantum optics, a superposition $\alpha_1\hat{\vec A}_1(\vec r,t)+\alpha_2\hat{\vec A}_2(\vec r,t)$ is generally **not** a valid solution, because it would alter the commutators; e.g. commutator (18). The field operator $\hat{\vec A}$ is already describing **all possible solutions** (for a given geometry and given boundary conditions). The specific realization of a superposition is given by the quantum state $\vert\Psi\rangle$.
Modes	We take a complete set of classical waves $\left\{\vec A_i(\vec r,t)\right\}$ (“mode functions”), which describe all possible solutions to Maxwell’s equations for a given geometry and given boundary conditions. For example, in free space ($\varepsilon=\text{const.},\ \mu=\text{const.}$), we can take the wave-vector $\vec k$ as index, and write $\boxed{\vec A_{\vec k}(\vec r,t)=\vec{\mathcal{A}}_{\vec k}(\vec r)\,e^{i\vec k\cdot\vec r-i\omega_kt}}$ ... (23) with $\omega_k=\frac{\lvert\vec k\rvert}{\sqrt{\varepsilon\mu}}$ and $\vec{\mathcal{A}}_{\vec k}(\vec r)$ being any spatial function. In general, the mode functions $\vec A_{\vec k}(\vec r,t)$ are complex-valued. For better readability, we will write $\vec A_k$ instead of $\vec A_{\vec k}$. We expand the Hermitian field operator into these modes: $\boxed{\hat{\vec A}(\vec r,t)=\sum_k\left(\vec A_k(\vec r,t)\,\hat a_k+\vec A_k^*(\vec r,t)\,\hat a_k^\dagger\right)}$ ... (24)
Normalization Orthogonality of Modes, Scalar Product	We impose that all mode functions are orthogonal and normalized. For this, we introduce a special scalar product: $\boxed{\left(\vec A_1,\vec A_2\right)\overset{\text{def}}{=}\frac{1}{i\hbar}\int\left(\vec A_1^*\cdot\vec D_2-\vec A_2\cdot\vec D_1^*\right)d^3r}$ ... $(25a)\overset{(4a)}{\Longrightarrow}\left(\vec A_1,\vec A_2\right)=\frac{\varepsilon_0\varepsilon}{i\hbar}\int\left(\vec A_1^*\cdot\vec E_2-\vec A_2\cdot\vec E_1^*\right)d^3r\overset{(4c)}{\Longrightarrow}$ $\boxed{\left(\vec A_1,\vec A_2\right)=\frac{i\varepsilon_0\varepsilon}{\hbar}\int\left(\vec A_1^*\cdot\partial_t\vec A_2-\vec A_2\cdot\partial_t\vec A_1^*\right)d^3r}$ ... $(25b)$ <u>not</u> positive definite: $\boxed{\left(\vec A_1^*,\vec A_2^*\right)=-\left(\vec A_1,\vec A_2\right)^*}$ ... $(25c)$ Linear in second entry: $\left(\vec A_0,\alpha_1\vec A_1+\alpha_2\vec A_2\right)=\alpha_1\left(\vec A_0,\vec A_1\right)+\alpha_2\left(\vec A_0,\vec A_2\right)$ Conjugate symmetric: $\boxed{\left(\vec A_1,\vec A_2\right)=\left(\vec A_2,\vec A_1\right)^*}$ ... $(25d)$ If orthogonal and normalized: $\boxed{\left(\vec A_k,\vec A_{k'}\right)=\delta_{kk'}}$ ... $(25e)\overset{(25c)}{\Longrightarrow}\boxed{\left(\vec A_k^*,\vec A_{k'}^*\right)=-\delta_{kk'}}$ ... $(25f)$ $\boxed{\left(\vec A_k,\vec A_{k'}^*\right)=0}$ ... $(25g)$ Note the similarity of $(25b)$ with the Klein-Gordon charge $Q=i\int(\psi^*\partial_t\psi-\psi\partial_t\psi^*)\,d^3r$
	Scalar product is conserved in time: $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{1}{i\hbar}\partial_t\int\left(\vec A_1^*\cdot\vec D_2-\vec A_2\cdot\vec D_1^*\right)d^3r=\frac{1}{i\hbar}\int\left(\partial_t\left(\vec A_1^*\cdot\vec D_2\right)-\partial_t\left(\vec A_2\cdot\vec D_1^*\right)\right)d^3r\Longrightarrow$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{1}{i\hbar}\int\left(\partial_t\vec A_1^*\cdot\vec D_2+\vec A_1^*\cdot\partial_t\vec D_2-\partial_t\vec A_2\cdot\vec D_1^*-\vec A_2\cdot\partial_t\vec D_1^*\right)d^3r\overset{(4c)}{=}\frac{1}{i\hbar}\int\left(-\vec E_1^*\cdot\vec D_2+\vec A_1^*\cdot\partial_t\vec D_2+\vec E_2\cdot\vec D_1^*-\vec A_2\cdot\partial_t\vec D_1^*\right)d^3r$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{i}{\hbar}\int\left(\vec E_1^*\cdot\vec D_2-\vec A_1^*\cdot\partial_t\vec D_2-\vec E_2\cdot\vec D_1^*+\vec A_2\cdot\partial_t\vec D_1^*\right)d^3r\overset{(3d)}{\Longrightarrow}$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{i}{\hbar}\int\left(\vec E_1^*\cdot\vec D_2-\vec A_1^*\cdot\left(\vec\nabla\times\vec H_2\right)-\vec E_2\cdot\vec D_1^*+\vec A_2\cdot\left(\vec\nabla\times\vec H_1^*\right)\right)d^3r$ ... $(26a)$ Auxillary calculation: $\vec\nabla\cdot\left(\vec A_1^*\times\vec H_2\right)=\vec H_2\cdot\left(\vec\nabla\times\vec A_1^*\right)-\vec A_1^*\cdot\left(\vec\nabla\times\vec H_2\right)\Longrightarrow$ $-\vec A_1^*\cdot\left(\vec\nabla\times\vec H_2\right)=\vec\nabla\cdot\left(\vec A_1^*\times\vec H_2\right)-\vec H_2\cdot\left(\vec\nabla\times\vec A_1^*\right)$ and analogously: $\vec A_2\cdot\left(\vec\nabla\times\vec H_1^*\right)=-\vec\nabla\cdot\left(\vec A_2\times\vec H_1^*\right)+\vec H_1^*\cdot\left(\vec\nabla\times\vec A_2\right)\overset{(26a)}{\Longrightarrow}$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{i}{\hbar}\int\left(\vec E_1^*\cdot\vec D_2+\vec\nabla\cdot\left(\vec A_1^*\times\vec H_2\right)-\vec H_2\cdot\left(\vec\nabla\times\vec A_1^*\right)-\vec E_2\cdot\vec D_1^*-\vec\nabla\cdot\left(\vec A_2\times\vec H_1^*\right)+\vec H_1^*\cdot\left(\vec\nabla\times\vec A_2\right)\right)d^3r\overset{(4d)}{\Longrightarrow}$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{i}{\hbar}\int\left(\vec E_1^*\cdot\vec D_2+\vec\nabla\cdot\left(\vec A_1^*\times\vec H_2\right)-\vec H_2\cdot\vec B_1^*-\vec E_2\cdot\vec D_1^*-\vec\nabla\cdot\left(\vec A_2\times\vec H_1^*\right)+\vec H_1^*\cdot\vec B_2\right)d^3r$ ... $(26b)$ $\vec E_1^*\cdot\vec D_2-\vec E_2\cdot\vec D_1^*\overset{(4a)}{=}\varepsilon_0\varepsilon\left(\vec E_1^*\cdot\vec E_2-\vec E_2\cdot\vec E_1^*\right)=0$ and $\vec H_1^*\cdot\vec B_2-\vec H_2\cdot\vec B_1^*\overset{(4b)}{=}\mu_0\mu\left(\vec H_1^*\cdot\vec H_2-\vec H_2\cdot\vec H_1^*\right)=0\overset{(26b)}{\Longrightarrow}$ $\partial_t\left(\vec A_1,\vec A_2\right)=\frac{i}{\hbar}\int\left(\vec\nabla\cdot\left(\vec A_1^*\times\vec H_2\right)-\vec\nabla\cdot\left(\vec A_2\times\vec H_1^*\right)\right)d^3r\overset{\text{Gauss}}{=}\frac{i}{\hbar}\oiint\left(\vec A_1^*\times\vec H_2-\vec A_2\times\vec H_1^*\right)d^2S=0\Longrightarrow\boxed{\partial_t\left(\vec A_1,\vec A_2\right)=0}$ ... (27)

Bose Commutators $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]$ $[\hat{a}_k\,,\hat{a}_{k'}]$ $[\hat{a}^\dagger_k\,,\hat{a}^\dagger_{k'}]$	From now on, we always consider normalized, orthogonal modes, fulfilling $(25efg)$. By projecting the respective mode $\vec{A}_k$ or $\vec{A}_k^*$ onto the expansion (24) of field operator $\hat{\vec{A}}$, one gets the coefficients $\hat{a}_k$ and $\hat{a}_k^\dagger$: $\left(\vec{A}_k\,,\hat{\vec{A}}\right)\overset{(24)}{=}\left(\vec{A}_k\,,\sum_{k'}(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\right)=\sum_{k'}\left((\vec{A}_k,\vec{A}_{k'})\hat{a}_{k'}+(\vec{A}_k,\vec{A}^*_{k'})\hat{a}^\dagger_{k'}\right)\overset{(25eg)}{=}\sum_{k'}\delta_{kk'}\hat{a}_{k'}\Rightarrow\boxed{\left(\vec{A}_k,\hat{\vec{A}}\right)=\hat{a}_k}\dots(28a)$ $\left(\vec{A}^*_k,\hat{\vec{A}}\right)\overset{(24)}{=}\left(\vec{A}^*_k,\sum_{k'}(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\right)=\sum_{k'}\left((\vec{A}^*_k,\vec{A}_{k'})\hat{a}_{k'}+(\vec{A}^*_k,\vec{A}^*_{k'})\hat{a}^\dagger_{k'}\right)\overset{(25fg)}{=}-\sum_{k'}\delta_{kk'}\hat{a}^\dagger_{k'}\Rightarrow\boxed{\left(\vec{A}^*_k,\hat{\vec{A}}\right)=-\hat{a}^\dagger_k}\dots(28b)$ $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]\overset{(28ab)}{=}\left[\left(\vec{A}_k\,,\hat{\vec{A}}\right),-\left(\vec{A}^*_k\,,\hat{\vec{A}}\right)\right]\xRightarrow{(25a)}$ $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]=\left[\frac{1}{i\hbar}\int\left(\vec{A}^*_k(\vec{r})\cdot\hat{\vec{D}}(\vec{r})-\hat{\vec{A}}(\vec{r})\cdot\vec{D}^*_k(\vec{r})\right)d^3r\;\;,\;-\frac{1}{i\hbar}\int\left(\vec{A}_{k'}(\vec{r}')\cdot\hat{\vec{D}}(\vec{r}')-\hat{\vec{A}}(\vec{r}')\cdot\vec{D}_{k'}(\vec{r}')\right)d^3r'\right]\Rightarrow$ $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]=\frac{1}{\hbar^2}\iint\left[\vec{A}^*_k(\vec{r})\cdot\hat{\vec{D}}(\vec{r})-\hat{\vec{A}}(\vec{r})\cdot\vec{D}^*_k(\vec{r})\;\;,\;\;\vec{A}_{k'}(\vec{r}')\cdot\hat{\vec{D}}(\vec{r}')-\hat{\vec{A}}(\vec{r}')\cdot\vec{D}_{k'}(\vec{r}')\right]d^3r\,d^3r'$ $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]=\frac{1}{\hbar^2}\iint\begin{pmatrix}\left(\vec{A}^*_k(\vec{r})\cdot\hat{\vec{D}}(\vec{r})-\hat{\vec{A}}(\vec{r})\cdot\vec{D}^*_{\vec{k}}(\vec{r})\right)\left(\vec{A}_{k'}(\vec{r}')\cdot\hat{\vec{D}}(\vec{r}')-\hat{\vec{A}}(\vec{r}')\cdot\vec{D}_{k'}(\vec{r}')\right)-\\\left(\vec{A}_{k'}(\vec{r}')\cdot\hat{\vec{D}}(\vec{r}')-\hat{\vec{A}}(\vec{r}')\cdot\vec{D}_{k'}(\vec{r}')\right)\left(\vec{A}^*_k(\vec{r})\cdot\hat{\vec{D}}(\vec{r})-\hat{\vec{A}}(\vec{r})\cdot\vec{D}^*_k(\vec{r})\right)\end{pmatrix}d^3r\,d^3r'$ $[\hat{a}_k\,,\hat{a}^\dagger_{k'}]=\frac{1}{i\hbar^2}\iint(\vec{A}^*_k(\vec{r})\cdot\vec{D}_{k'}(\vec{r})-\vec{A}_{k'}(\vec{r})\cdot\vec{D}^*_k(\vec{r}))\,d^3r\overset{(25a)}{=}(\vec{A}_k\,,\vec{A}_{k'})\xRightarrow{(25e)}\boxed{[\hat{a}_k\,,\hat{a}^\dagger_{k'}]=\delta_{kk'}}\dots(29a)$ Analogously: $\boxed{[\hat{a}_k\,,\hat{a}_{k'}]=0}\dots(29b)\quad\Leftrightarrow\quad\boxed{[\hat{a}^\dagger_k\,,\hat{a}^\dagger_{k'}]=0}\dots(29c)$
Vacuum state	Vacuum state $\vert 0\rangle$ defined by $\boxed{\hat{a}_k\vert 0\rangle=0}\dots(30)$ $\hat{a}_k$ … annihilation operator $\hat{a}^\dagger_k$…creation operator
Interferences	Two Types: (a) Quantum interference / superposition: $\vert\psi\rangle=\alpha_1\vert\psi_1\rangle+\alpha_2\vert\psi_1\rangle$ takes place in Hilbert space $\mathcal{H}$ (b) optical interference $\vec{A}=\alpha_1\vec{A}_1+\alpha_2\vec{A}_2$ takes place in mode space(space of all solutions to the wave equation)
Monochromatic modes	Generally, modes can be pulses. Important special case: Monochromatic modes, defined by $\boxed{\partial_t\vec{A}_k=-i\omega_k\vec{A}_k\Leftrightarrow\partial_t\vec{A}^*_k=i\omega_k\vec{A}^*_k}\dots(31a)\Leftrightarrow\boxed{\vec{A}_k(\vec{r},t)=\vec{A}_k(\vec{r})\,e^{-i\omega_k t}}\dots(31b)$ Scalar product of monochr. modes: $\left(\vec{A}_1,\vec{A}_2\right)\overset{(25b)}{=}\frac{i\varepsilon_0\varepsilon}{\hbar}\int(\vec{A}^*_1\cdot\partial_t\vec{A}_2-\vec{A}_2\cdot\partial_t\vec{A}^*_1)\,d^3r\overset{(31b)}{=}\frac{i\varepsilon_0\varepsilon}{\hbar}\int\left(\vec{A}^*_1e^{i\omega_1t}\cdot\partial_t\left(\vec{A}_2e^{-i\omega_2t}\right)-\vec{A}_2e^{-i\omega_2t}\cdot\partial_t\left(\vec{A}^*_1e^{i\omega_1t}\right)\right)d^3r\Rightarrow$ $\left(\vec{A}_1,\vec{A}_2\right)=\frac{i\varepsilon_0\varepsilon}{\hbar}\int(-i\omega_2e^{i\omega_1t}e^{-i\omega_2t}\,\vec{A}^*_1\cdot\vec{A}_2-i\omega_1e^{-i\omega_2t}e^{i\omega_1t}\,\vec{A}_2\cdot\vec{A}^*_1)\,d^3r\Rightarrow$ $\left(\vec{A}_1,\vec{A}_2\right)=\frac{\varepsilon_0\varepsilon}{\hbar}\int(\omega_1e^{i(\omega_1-\omega_2)t}\;+\omega_2e^{i(\omega_1-\omega_2)t}\;)\vec{A}^*_1\cdot\vec{A}_2\,d^3r\Rightarrow\boxed{\left(\vec{A}_1,\vec{A}_2\right)=\frac{\varepsilon_0\varepsilon(\omega_1+\omega_2)}{\hbar}e^{i(\omega_1-\omega_2)t}\int\vec{A}^*_1\cdot\vec{A}_2\,d^3r}\dots(32a)$ $\boxed{\text{if }\;\omega_1=\omega_2=\omega_k\Rightarrow\left(\vec{A}_1,\vec{A}_2\right)=\frac{\varepsilon_0\varepsilon2\omega_k}{\hbar}\int\vec{A}^*_1\cdot\vec{A}_2\,d^3r}\dots(32b)$
Energy (only for monochromatic waves)	$(10a)\Rightarrow\hat{H}=\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{B}}\cdot\hat{\vec{H}}\right)dV\overset{(7b)}{=}\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\left(\vec{\nabla}\times\hat{\vec{A}}\right)\cdot\hat{\vec{H}}\right)dV\,\Big\vert\,\left(\vec{\nabla}\times\hat{\vec{A}}\right)\cdot\hat{\vec{H}}=\hat{\vec{A}}\left(\vec{\nabla}\times\hat{\vec{H}}\right)+\vec{\nabla}\cdot\left(\hat{\vec{A}}\times\hat{\vec{H}}\right)$ $\hat{H}=\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{A}}\cdot\left(\vec{\nabla}\times\hat{\vec{H}}\right)+\vec{\nabla}\cdot\left(\hat{\vec{A}}\times\hat{\vec{H}}\right)\right)dV\overset{(5d)}{=}\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}+\vec{\nabla}\cdot\left(\hat{\vec{A}}\times\hat{\vec{H}}\right)\right)dV\Rightarrow$ $\hat{H}=\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}\right)dV+\frac{1}{2}\int\vec{\nabla}\cdot\left(\hat{\vec{A}}\times\hat{\vec{H}}\right)dV\overset{\text{Gauss}}{=}\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}\right)dV+\cancel{\frac{1}{2}\oint\hat{\vec{A}}\times\hat{\vec{H}}\,d^2S}\Rightarrow$ $\boxed{\hat{H}=\frac{1}{2}\int\left(\hat{\vec{E}}\cdot\hat{\vec{D}}+\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}\right)dV}\dots(33)$ $\hat{\vec{E}}\cdot\hat{\vec{D}}\overset{(6a)}{=}\varepsilon_0\varepsilon\,\hat{\vec{E}}\cdot\hat{\vec{E}}\overset{(7a)}{=}\varepsilon_0\varepsilon\left(\partial_t\hat{\vec{A}}\right)\cdot\left(\partial_t\hat{\vec{A}}\right)\overset{(24)}{=}\varepsilon_0\varepsilon(\partial_t\sum_k(\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k))\cdot(\partial_t\sum_{k'}(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'}))\xRightarrow{(31a)}$ $\hat{\vec{E}}\cdot\hat{\vec{D}}=\varepsilon_0\varepsilon(\sum_k(-i\omega_k\vec{A}_k\hat{a}_k+i\omega_k\vec{A}^*_k\hat{a}^\dagger_k))\cdot(\sum_{k'}(-i\omega_{k'}\vec{A}_{k'}\hat{a}_{\vec{k}'}+i\omega_{k'}\vec{A}^*_{k'}\hat{a}^\dagger_{k'}))\Rightarrow$ $\hat{\vec{E}}\cdot\hat{\vec{D}}=\varepsilon_0\varepsilon\sum_{kk'}i\omega_k(-\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k)\cdot i\omega_{k'}(-\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\Rightarrow$ $\hat{\vec{E}}\cdot\hat{\vec{D}}=-\varepsilon_0\varepsilon\sum_{kk'}\omega_k\omega_{k'}(\vec{A}_k\hat{a}_k-\vec{A}^*_k\hat{a}^\dagger_k)\cdot(\vec{A}_{k'}\hat{a}_{k'}-\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\dots(34a)$ $\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}\overset{(6a)}{=}\varepsilon_0\varepsilon\hat{\vec{A}}\cdot\partial_t\hat{\vec{E}}\overset{(7a)}{=}-\varepsilon_0\varepsilon\hat{\vec{A}}\cdot\partial_t^2\hat{\vec{A}}\overset{(24)}{=}-\varepsilon_0\varepsilon(\sum_k(\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k))\cdot(\partial_t^2\sum_{k'}(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'}))\xRightarrow{(31a)}$ $\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}=-\varepsilon_0\varepsilon(\sum_{\vec{k}}(\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k))\cdot(\sum_{k'}(-\omega^2_{k'}\vec{A}_{k'}\hat{a}_{k'}-\omega^2_{k'}\vec{A}^*_{k'}\hat{a}^\dagger_{k'}))\Rightarrow$ $\hat{\vec{A}}\cdot\partial_t\hat{\vec{D}}=\varepsilon_0\varepsilon\sum_{kk'}\omega^2_{k'}(\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k)\cdot(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\dots(34b)$ $\left.\begin{matrix}(34a)\\(34b)\end{matrix}\right\}\xRightarrow{(33)}$ $\hat{H}=-\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega_k\omega_{k'}\int(\vec{A}_k\hat{a}_k-\vec{A}^*_k\hat{a}^\dagger_k)\cdot(\vec{A}_{k'}\hat{a}_{k'}-\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\,dV+\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega^2_{k'}\int(\vec{A}_k\hat{a}_k+\vec{A}^*_k\hat{a}^\dagger_k)\cdot(\vec{A}_{k'}\hat{a}_{k'}+\vec{A}^*_{k'}\hat{a}^\dagger_{k'})\,dV$ $\hat{H}=\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega_k\omega_{k'}\int(-\vec{A}_k\cdot\vec{A}_{k'}\hat{a}_k\hat{a}_{k'}+\vec{A}^*_k\cdot\vec{A}_{k'}\hat{a}^\dagger_k\hat{a}_{k'}+\vec{A}_k\cdot\vec{A}^*_{k'}\hat{a}_k\hat{a}^\dagger_{k'}-\vec{A}^*_k\cdot\vec{A}^*_{k'}\hat{a}^\dagger_k\hat{a}^\dagger_{k'})\,dV$ $\quad+\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega^2_{k'}\int(\vec{A}_k\cdot\vec{A}_{k'}\hat{a}_k\hat{a}_{k'}+\vec{A}^*_k\cdot\vec{A}_{k'}\hat{a}^\dagger_k\hat{a}_{k'}+\vec{A}_k\cdot\vec{A}^*_{k'}\hat{a}_k\hat{a}^\dagger_{k'}+\vec{A}^*_k\cdot\vec{A}^*_{k'}\hat{a}^\dagger_k\hat{a}^\dagger_{k'})\,dV\dots(35a)$ $(32b)\Rightarrow\int\vec{A}_k\cdot\vec{A}_{k'}\,dV=\frac{\hbar}{2\varepsilon_0\varepsilon\omega_k}(\vec{A}^*_k\,,\vec{A}_{k'})\overset{(25g)}{=}0$, analogously $\int\vec{A}^*_k\cdot\vec{A}^*_{k'}\,dV=0\xRightarrow{(35a)}$ $\hat{H}=\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega_k\omega_{k'}\int(\vec{A}^*_k\cdot\vec{A}_{k'}\hat{a}^\dagger_k\hat{a}_{k'}+\vec{A}_k\cdot\vec{A}^*_{k'}\hat{a}_k\hat{a}^\dagger_{k'})\,dV+\frac{1}{2}\varepsilon_0\varepsilon\sum_{kk'}\omega^2_{k'}\int(\vec{A}^*_k\cdot\vec{A}_{k'}\hat{a}^\dagger_k\hat{a}_{k'}+\vec{A}_k\cdot\vec{A}^*_{k'}\hat{a}_k\hat{a}^\dagger_{k'})\,dV\dots(35b)$ $(32b)\Rightarrow\int\vec{A}^*_k\cdot\vec{A}_{k'}\,dV=\frac{\hbar}{2\varepsilon_0\varepsilon\omega_k}(\vec{A}_k\,,\vec{A}_{k'})\overset{(7a)}{=}\frac{\hbar}{2\varepsilon_0\varepsilon\omega_k}\delta_{kk'}$, analogously $\int\vec{A}_k\cdot\vec{A}^*_{k'}\,dV=\frac{\hbar}{2\varepsilon_0\varepsilon\omega_k}\delta_{kk'}\xRightarrow{(35b)}$ $\hat{H}=\frac{1}{2}\sum_{kk'}\int\left(\omega_k\omega_{k'}\frac{\hbar}{2\omega_k}\delta_{kk'}\hat{a}^\dagger_k\hat{a}_{k'}+\omega_k\omega_{k'}\frac{\hbar}{2\omega_k}\delta_{kk'}\hat{a}_k\hat{a}^\dagger_{k'}\right)dV+\frac{1}{2}\sum_{kk'}\int\left(\omega^2_{k'}\frac{\hbar}{2\omega_k}\delta_{kk'}\hat{a}^\dagger_k\hat{a}_{k'}+\omega^2_{k'}\frac{\hbar}{2\omega_k}\delta_{kk'}\hat{a}_k\hat{a}^\dagger_{k'}\right)dV$ $\hat{H}=\frac{1}{2}\sum_k\int\left(\frac{\hbar\omega_k}{2}\hat{a}^\dagger_k\hat{a}_k+\frac{\hbar\omega_k}{2}\hat{a}_k\hat{a}^\dagger_k+\frac{\hbar\omega_k}{2}\hat{a}^\dagger_k\hat{a}_k+\frac{\hbar\omega_k}{2}\hat{a}_k\hat{a}^\dagger_k\right)dV\Rightarrow\hat{H}=\sum_k\frac{\hbar\omega_k}{2}(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k)\dots(35c)$ $(39a)\Rightarrow[\hat{a}_k,\hat{a}^\dagger_k]=1\Rightarrow\hat{a}_k\hat{a}^\dagger_k-\hat{a}^\dagger_k\hat{a}_k=1\Rightarrow\hat{a}_k\hat{a}^\dagger_k=1+\hat{a}^\dagger_k\hat{a}_k\xRightarrow{(35c)}\hat{H}=\sum_k\frac{\hbar\omega_k}{2}(1+\hat{a}^\dagger_k\hat{a}_k+\hat{a}^\dagger_k\hat{a}_k)\Rightarrow$ $\boxed{\hat{H}=\sum_k\hbar\omega_k\left(\hat{a}^\dagger_k\hat{a}_k+\frac{1}{2}\right)}\dots(36)$
Vacuum Energy	$E_0=\langle 0\vert\hat{H}\vert 0\rangle\overset{(36)}{=}\sum_k\hbar\omega_k\left(\cancel{\langle 0\vert\hat{a}^\dagger_k\hat{a}_k\vert 0\rangle}+\frac{1}{2}\langle 0\vert 0\rangle\right)\Rightarrow\boxed{E_0=\sum_k\frac{\hbar\omega_k}{2}\rightarrow\infty}\dots(37)$ The vacuum energy is infinite! Even a cutoff at plank length does not solve the problem, as the corresponding mass-density would be so huge that the whole universe would immediately collapse into a black hole. But the result cannot be ignored, as it leads by means of renormalization to observable physics, e.g. vacuum forces or the Casimir effect.

2 Casimir Effect

2.1 Summing up Zero-Point Energies

1D-example	1D configuration with two plane parallel mirrors at distance a, the field vanishes at the mirrors. Mode frequencies: $f_m = \frac{c}{\lambda_m} = \frac{c}{\lambda_0/m} = \frac{c}{\lambda_0} m = \frac{c}{2a} m$ ... (38) $\omega_m = 2\pi f_m \overset{(38)}{\Longrightarrow} \omega_m = c\frac{\pi}{a} m$ with $m \in \mathbb{N}$ ... (39) $\overset{(37)}{\Longrightarrow}$ $\boxed{E_0 = \frac{\hbar c}{2a}\pi \sum_m m}$ ... (40)
Ramanujan sum	$\boxed{\sum_m m = -\frac{1}{12}}$ ... (41) Hand-wavy proof: $\left.\begin{array}{l} S = 1+2+3+4+5+6+\cdots \\ \underline{4S = \quad 4 \quad + \quad 8 \quad + \quad 12 + \cdots} \end{array}\right\} -$ $-3S = 1-2+3-4+5-6+\cdots$ $\begin{array}{ccccc} & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & +1 & -1 \\ -1 & -1 & +1 & -1 & \\ +1 & +1 & -1 & \ddots & \\ -1 & -1 & & & \end{array}$ $\left(\sum_{m=0}^{\infty} z^m \big\vert_{z=-1}\right)^2 =$ $(1-1+1-1+\cdots)^2 =$ $1-2+3-4+\cdots = -3S =$ $\left(\frac{1}{1-z}\Big\vert_{z=-1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Longrightarrow S = -\frac{1}{12}$ ✓
Zeta function renormalization	Riemann Zeta Function $\zeta(x) = \sum_{m=1}^{\infty} \frac{1}{m^x}$ ... (42) $\sum_{m=1}^{\infty} m = \sum_{m=1}^{\infty} \frac{1}{m^{-1}} = \zeta(-1) \Longrightarrow$ not defined with def. (42), but analytical continuation of $\zeta(x)$ gives $\zeta(-1) = -\frac{1}{12}$ ✓
Euler-Maclaurin	$\int_0^{\infty} f(m)\, dm = \sum_{m=1}^{\infty} f(m) + \frac{f(0)}{2} + \frac{f'(0)}{12} - \frac{f'''(0)}{720} + \cdots \Big\vert\, f(m) = m \Longrightarrow f(0) = 0,\ f'(0) = 1, f'''(0) = 0 \ldots \Longrightarrow$ $\int_0^{\infty} m\, dm = \sum_{m=1}^{\infty} m + \frac{1}{12} \Longrightarrow \sum_{m=1}^{\infty} m = -\frac{1}{12} + \int_0^{\infty} m\, dm$ ... (43) then, drop the divergent integral.
Casimir force 1D	(41) in (40) $\Longrightarrow$ $\boxed{E_0^{1D} = -\frac{\pi}{24}\frac{\hbar c}{a}}$ ... (44) $F_{1D} = -\frac{\partial E_0^{1D}}{\partial a} \overset{(44)}{\Longrightarrow}$ $\boxed{F_{1D} = -\frac{\pi}{24}\frac{\hbar c}{a^2}}$ ... (45) force acting on right mirror
Casimir force 3D	2 plane parallel quadratic plates, distance a, side length $L \gg a$: $\boxed{E_0^{3D} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3} L^2}$ ... (46) Energy density: $\epsilon = \frac{E_0^{3D}}{V} = \frac{E_0^{3D}}{L^2 a} \overset{(46)}{\Longrightarrow}$ $\boxed{\epsilon = \frac{\pi^2}{720}\frac{\hbar c}{a^4}}$ ... (47) $F_{3D} = -\frac{\partial E_0^{3D}}{\partial a} \overset{(45)}{\Longrightarrow}$ $\boxed{F_{3D} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} L^2}$ ... (48) Force density: $f = \frac{F_{3D}}{L^2} \overset{(45)}{\Longrightarrow}$ $\boxed{f = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}}$ ... (49)
Point Particle Theory	Theory: Why is charged point particle stable? Equilibrium between outgoing repulsive Electrostatic force and ingoing Casimir force! Energy-equilibrium equation: $C\frac{\hbar c}{a^4} = \frac{e^2}{a^4}$ with C ...geometry constant. Not dependent on a! Particle could become arbitrarily small! Note: $\frac{1}{C} = \frac{e^2}{\hbar c} = \alpha \approx \frac{1}{137}$ But: Theory is wrong. Casimir Force in that case is repulsive!
Solid Sphere	Unsolved, except for dispersion-less media.

2.2 Lifshitz Theory

Flucutation Dissipation Theorem	Consider 1D EM Potential: $(4c) \Longrightarrow E = -\partial_t A$ ... $(50a)$ $(4d) \Longrightarrow B = \partial_x A$ ... $(50b)$ Energy Density: $\epsilon = \frac{1}{2}(ED + BH) \overset{(4ab)}{=} \frac{1}{2}\left(\varepsilon_0 \varepsilon E^2 + \frac{B}{\mu_0 \mu}\frac{\varepsilon_0}{\varepsilon_0}\right) = \frac{\varepsilon_0}{2}\left(\varepsilon E^2 + \frac{B^2}{\mu}\frac{1}{\mu_0 \varepsilon_0}\right) \overset{(6c)}{=} \frac{\varepsilon_0}{2}\left(\varepsilon E^2 + \frac{c^2 B^2}{\mu}\right) \overset{(50ab)}{=} \frac{\varepsilon_0}{2}\left(\varepsilon \partial_t \partial_t + \frac{c^2}{\mu}\partial_x \partial_x\right) A$ ... (51) Point-splitting: $A = \frac{1}{2}\langle \hat{A}(x_1, t_1)\, \hat{A}(x_2, t_2) + \hat{A}(x_2, t_2)\, \hat{A}(x_1, t_1)\rangle\Big\vert_{t_1 = t_2,\ x_1 \to x_2}$ $\in \mathbb{R}$ (because symmetric $\Longrightarrow$hermitian) $\overset{(51)}{\Longrightarrow}$ $\epsilon = \frac{1}{2}\frac{\hbar}{c}\left(\varepsilon \partial_{t_1} \partial_{t_2} + \frac{c^2}{\mu}\partial_{x_1}\partial_{x_2}\right) \varepsilon_0 \frac{c}{\hbar}\frac{1}{2}\langle \hat{A}(x_1 t_1)\, \hat{A}(x_2 t_2) + \hat{A}(x_2 t_2)\, \hat{A}(x_1 t_1)\rangle\Big\vert_{t_1 = t_2,\ x_1 \to x_2}$ $\Longrightarrow$ $\boxed{\epsilon = \frac{\hbar}{2c}\left(\varepsilon \partial_{t_1} \partial_{t_2} + \frac{c^2}{\mu}\partial_{x_1}\partial_{x_2}\right) K\big\vert_{t_1 = t_2,\ x_1 \to x_2}}$ ... (52) with $\boxed{K \overset{\text{def}}{=} \frac{\varepsilon_0 c}{2\hbar}\langle \hat{A}_1 \hat{A}_2 + \hat{A}_2 \hat{A}_1\rangle \text{ with } \hat{A}_1 = \hat{A}(x_1, t_1),\ \hat{A}_2 = \hat{A}(x_2, t_1)}$ ... (53) correlation function $\boxed{\Gamma \overset{\text{def}}{=} \frac{\varepsilon_0 c}{2i\hbar}\langle \hat{A}_2 \hat{A}_1 - \hat{A}_1 \hat{A}_2\rangle = \frac{1}{2c}(G_+ - G_-) \in \mathbb{R} \text{ with } \hat{A}_1 = \hat{A}(x_1 t_1),\ \hat{A}_2 = \hat{A}(x_2, t_1)}$ ... (54) Retarded Green's Function: $\boxed{G_+ = \frac{\varepsilon_0 c^2}{i\hbar}\Theta(t_2 - t_1)\langle[\hat{A}_2, \hat{A}_1]\rangle}$ ... $(55a)$ for $t_2 > t_1$... emission of em radiation, "switched on" Advanced Green's Function: $\boxed{G_- = \frac{\varepsilon_0 c^2}{i\hbar}\Theta(t_1 - t_2)\langle[\hat{A}_2, \hat{A}_1]\rangle}$ ... $(55b)$ for $t_2 < t_1$... incoming and absorbed em radiation $\boxed{[\hat{A}(x_2, t), \hat{D}(x_1, t)] = i\hbar\, \delta(x_1 - x_2)}$ ... (57) $\Longrightarrow$ $\boxed{\left(\partial_x \frac{1}{\mu}\partial_x - \frac{\varepsilon}{c^2}\partial_t^2\right) G_\pm = \delta(t_1 - t_2)\, \delta(x_1 - x_2)}$ ... (58)
Kramers Kronig	$\boxed{K = \mathrm{Re}(f), \Gamma = \mathrm{Im}(f)}$ ... $(59a)$ with $\boxed{f(t) = \frac{\varepsilon_0 c}{\hbar}\langle \hat{A}_2 \hat{A}_1\rangle \text{ and } t = t_2 - t_1}$ ... $(59b)$ Correlation K is related to Γ, which is the difference between advanced and retarded Green's function (i.e. the dissipation) $\mathrm{Re}(f) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(f(t'))}{t' - t} dt' \overset{(59ab)}{\Longrightarrow}$ $\boxed{K = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Gamma(t')}{t' - t} dt'}$ ... $(59c)$ with $\mathcal{P}$... Principal value
Fourier, Convolution,	Fourier Transform: $\mathcal{F}[f(t)] = \tilde{f}(\omega) \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$ ... $(60a)$ Inverse Fourier Transform: $\mathcal{F}^{-1}[\tilde{f}(\omega)] = f(t) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega$ ... $(60b)$ Convolution: $f_1 * f_2 = \mathcal{F}^{-1}[\tilde{f}_1 \tilde{f}_2] \Longrightarrow \int_{-\infty}^{\infty} f_1(t - t')\, f_2(t')\, dt' = \int_{-\infty}^{\infty} f_1(t)\, f_2(t - t')\, dt' = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_1(\omega)\, \tilde{f}_2(\omega)\, e^{-i\omega t}\, d\omega$ ... $(60c)$ $\mathcal{F}\left[\frac{1}{i\pi t}\right] = \mathrm{sgn}(\omega)$ ... $(60d)$ $\Leftrightarrow$ $\boxed{\mathcal{F}\left[\frac{1}{t}\right] = i\pi\, \mathrm{sgn}(\omega)}$ ... $(60e)$ because $\mathcal{F}^{-1}[\mathrm{sgn}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{sgn}(\omega)\, e^{-i\omega t}\, d\omega = -\frac{1}{2\pi}\int_{-\infty}^{0} e^{-i\omega t}\, d\omega + \frac{1}{2\pi}\int_0^{\infty} e^{-i\omega t}\, d\omega = -\left(-\frac{1}{2\pi i t} + \delta(t)\right) + \frac{1}{2\pi i t} + \delta(t) = \frac{1}{i\pi t}$ ✓ $\boxed{\tilde{G}_+ = \tilde{G}_-^* \overset{\text{def}}{=} \tilde{G} \Leftrightarrow \tilde{G}_- = \tilde{G}^*}$ ... $(60f)$ $G \in \mathbb{R} \Longrightarrow$ $\boxed{\tilde{G}(-\omega) = \tilde{G}^*(\omega) \overset{(60f)}{=} \tilde{G}_-(\omega)}$ ... $(60g)$

Fluctuation Correlation Function K

To be clarified: Sign should be minus!

$\boxed{\tilde{\Gamma}(\omega) \overset{(54)}{=} \frac{1}{2c}\left(\tilde{G}_+ - \tilde{G}_-\right)}$ ... $(60h) \overset{(60f)}{\Longrightarrow} \tilde{\Gamma}(\omega) = \frac{i}{c}\frac{1}{2i}\left(\tilde{G} - \tilde{G}^*\right) \Longrightarrow \boxed{\tilde{\Gamma}(\omega) = \frac{i}{c}\operatorname{Im}(\tilde{G})}$ ... $(61a) \Leftrightarrow \boxed{\operatorname{Im}(\tilde{G}) = \frac{c}{i}\tilde{\Gamma}(\omega)}$ ... $(61b)$

$(59c) \Longrightarrow K = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{t'-t}\Gamma(t')\,dt' = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{t-t'}\Gamma(t')\,dt' = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{t'}\Gamma(t-t')\,dt' \overset{(60c)}{\Longrightarrow}$

$K = -\frac{1}{2\pi}\frac{1}{\pi}\int_{-\infty}^{\infty}\mathcal{F}\left[\frac{1}{t}\right]\tilde{\Gamma}(\omega)\,e^{-i\omega t}\,d\omega \overset{(60e)}{=} -\frac{i}{2\pi}\int_{-\infty}^{\infty}\operatorname{sgn}(\omega)\,\tilde{\Gamma}(\omega)\,e^{-i\omega t}\,d\omega \Rightarrow \boxed{K \overset{?}{=} -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\operatorname{sgn}(\omega)\,\tilde{\Gamma}(\omega)\,e^{-i\omega t}\,d\omega}$ ... $(62) \overset{(61a)}{\Longrightarrow}$

$K = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\operatorname{sgn}(\omega)\operatorname{Im}(\tilde{G})\,e^{-i\omega t}\,d\omega = \frac{1}{2\pi c}\left(-\int_{-\infty}^{0}\operatorname{Im}(\tilde{G}(\omega'))e^{-i\omega' t}\,d\omega' + \int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,e^{-i\omega t}\,d\omega\right)\Bigg|\begin{matrix}\omega' = -\omega \Longrightarrow -\infty \to \infty\\ d\omega' = -d\omega\end{matrix}$

$K = \frac{1}{2\pi c}\left(\int_{\infty}^{0}\operatorname{Im}(\tilde{G}(-\omega))\,e^{i\omega t}\,d\omega + \int_{0}^{\infty}\operatorname{Im}(\tilde{G}(\omega))\,e^{-i\omega t}\,d\omega\right)\Big|\operatorname{Im}(\tilde{G}(-\omega)) \overset{(60g)}{=} \operatorname{Im}(\tilde{G}^*(\omega)) = -\operatorname{Im}(\tilde{G}(\omega)) \equiv -\operatorname{Im}(\tilde{G})$

$K = \frac{1}{2\pi c}\left(-\int_{\infty}^{0}\operatorname{Im}(\tilde{G})\,e^{i\omega t}\,d\omega + \int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,e^{-i\omega t}\,d\omega\right) = \frac{1}{2\pi c}\left(\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,e^{i\omega t}\,d\omega + \int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,e^{-i\omega t}\,d\omega\right) \Longrightarrow$

$K = \frac{1}{2\pi c}\int_{0}^{\infty}\left(\operatorname{Im}(\tilde{G})\,e^{i\omega t} + \operatorname{Im}(\tilde{G})\,e^{-i\omega t}\right)d\omega = \frac{i}{\pi c}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)d\omega \Rightarrow \boxed{K \overset{?}{=} -\frac{1}{\pi c}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t)\,d\omega}$ ...(63)

$\operatorname{Im}(\tilde{G})$ describes dissipation (losses or gain). Even if there are no losses, but a redistribution of Fourier components, $\operatorname{Im}(\tilde{G}) \neq 0$

Energy Density

$\overset{(52)}{\Longrightarrow} \epsilon = -\frac{\hbar}{2\pi c^2}\left(\varepsilon\partial_{t_1}\partial_{t_2} + \frac{c^2}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t)\,d\omega\,\Big|\, t \overset{\text{def}}{=} t_1 - t_2$

$\epsilon = -\frac{\hbar}{2\pi c^2}\left(\varepsilon\partial_{t_1}\partial_{t_2} + \frac{c^2}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t_1 - \omega t_2)\,d\omega$

$\epsilon = -\frac{\hbar}{2\pi}\left(\varepsilon\frac{1}{c^2}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,\partial_{t_1}\partial_{t_2}\cos(\omega t_1 - \omega t_2)\,d\omega - \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t_1 - \omega t_2)\,d\omega\right)$ ... (64)

$\partial_{t_1}\partial_{t_2}\cos(\omega t_1 - \omega t_2) = -\omega\partial_{t_1}(-\sin(\omega t_1 - \omega t_2)) = \omega^2\cos(\omega t_1 - \omega t_2) \overset{(64)}{\Longrightarrow}$

$\epsilon = -\frac{\hbar}{2\pi}\left(\varepsilon\frac{\omega^2}{c^2}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t_1 - \omega t_2)\,d\omega - \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t_1 - \omega t_2)\,d\omega\right)$

$\epsilon = -\frac{\hbar}{2\pi}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\cos(\omega t_1 - \omega t_2)\,d\omega\,\Big|\, t_1 = t_2 \Longrightarrow \epsilon = \frac{\hbar}{2\pi}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\operatorname{Im}(\tilde{G})\,d\omega \overset{(61b)}{\Longrightarrow}$

$\epsilon = -\frac{\hbar c}{2\pi i}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\tilde{\Gamma}(\omega)\,d\omega \overset{(54)}{=} \frac{\hbar}{4\pi i}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\int_{0}^{\infty}\left(\tilde{G}_+ - \tilde{G}_-\right)d\omega \overset{(60f)}{\Longrightarrow}$

$\boxed{\epsilon = -\frac{\hbar}{4\pi i}\int_{0}^{\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\left(\tilde{G} - \tilde{G}^*\right)d\omega}$ ... $(65) \overset{(60g)}{\Longrightarrow}$

Lifshitz Formula

$\epsilon = -\frac{\hbar}{4\pi i}\int_{0}^{\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\left(\tilde{G}(\omega) - \tilde{G}(-\omega)\right)d\omega \Longrightarrow$

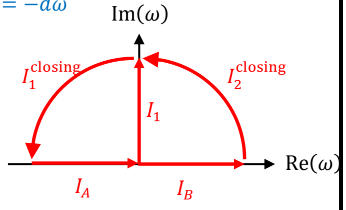
$\epsilon = -\frac{\hbar}{4\pi i}\left(\int_{0}^{\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega - \int_{0}^{\infty}\left(\varepsilon\frac{\omega'^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(-\omega')\,d\omega'\right)\Bigg|\begin{matrix}\omega' = -\omega \Longrightarrow \omega' = \infty \to \omega = -\infty\\ d\omega' = -d\omega\end{matrix}$

$\epsilon = -\frac{\hbar}{4\pi i}\left(\int_{0}^{\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega + \int_{0}^{-\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega\right) \Longrightarrow$

$\epsilon = -\frac{\hbar}{4\pi i}\left(\int_{0}^{\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega - \int_{-\infty}^{0}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega\right)$

$\epsilon \overset{\text{def}}{=} -\frac{\hbar}{4\pi i}\left(I_B - I_A\right) \overset{(66ab)}{\Longrightarrow}$ $\qquad I_A + I_1 + \cancel{I_1^{\text{closing}}} = 0 \Longrightarrow I_A = -I_1$ ... $(66a)$

$\epsilon = -\frac{\hbar}{4\pi i}\left(I_1 + I_1\right) = -\frac{\hbar}{2\pi i}I_1$ ... $(66c)$ $\qquad I_B + \cancel{I_2^{\text{closing}}} - I_1 = 0 \Longrightarrow I_B = \ \ I_1$ ... $(66b)$



$I_1 = \int_{0}^{i\infty}\left(\varepsilon\frac{\omega^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(\omega)\,d\omega\,\Big|\,\omega = i\xi \Longrightarrow \xi = \frac{\omega}{i} \Longrightarrow \omega = i\infty \to \xi = \infty;\ \frac{d\omega}{d\xi} = i \Longrightarrow d\omega = id\xi$

$I_1 = i\int_{0}^{\infty}\left(-\varepsilon\frac{\xi^2}{c^2} + \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\,d\xi \overset{(66c)}{\Longrightarrow} \epsilon = \frac{\hbar}{2\pi}\int_{0}^{\infty}\left(\varepsilon\frac{\xi^2}{c^2} - \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\,d\xi \Longrightarrow$

$\boxed{\epsilon = \frac{\hbar}{2\pi}\int_{0}^{\infty}\left(\varepsilon\frac{\xi^2}{c^2} - \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\,d\xi}$ ... (67)

$\varepsilon(i\xi) \in \mathbb{R}$

Note: with $\omega \in \mathbb{R}$: $\varepsilon(\omega) \in \mathbb{C}$, but because $\boxed{\varepsilon^*(\omega) = \varepsilon(-\omega)}$ ... $(68a) \Longrightarrow \boxed{\varepsilon(i\xi), \mu(i\xi) \in \mathbb{R}}$ ... $(68b)$

Damped harmonic oscillator:

$\varepsilon(i\xi)$ smoothly decaying

$m\ddot{x} = -b\dot{x} - Dx + \mathrm{F}(t) \Longrightarrow m\ddot{x} + b\dot{x} + Dx = F \Longrightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{D}{m}x = \frac{F}{m}\Big|\frac{b}{m} \overset{\text{def}}{=} \gamma, \frac{D}{m} \overset{\text{def}}{=} \omega_0^2 \Longrightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m} \overset{FT}{\Rightarrow}$

$(-i\omega)\tilde{x} + \gamma(-i\omega)\tilde{x} + \omega_0^2\tilde{x} = \frac{\tilde{F}}{m} \Longrightarrow -\omega^2\tilde{x} - i\gamma\omega\tilde{x} + \omega_0^2\tilde{x} = \frac{\tilde{F}}{m} \Longrightarrow \tilde{x}(\omega_0^2 - \omega^2 - i\gamma\omega) = \frac{\tilde{F}}{m} \Longrightarrow \boxed{\tilde{x} = \frac{\tilde{F}/m}{\omega_0^2 - \omega^2 - i\gamma\omega}}$ ... $(69a)$

Electrical Polarization: $P = \int_{-\infty}^{\infty}\chi_e(t - t')\,E(t')\,dt \overset{(60c)}{\Longrightarrow} \tilde{P} = \tilde{\chi}_e\tilde{E} \Longrightarrow \tilde{\chi}_e = \frac{\tilde{P}}{\tilde{E}}$ ... $(69b)$

$\tilde{P} = e\tilde{x} \overset{(69a)}{=} \frac{e\tilde{F}/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\Big|\tilde{F} = e\tilde{E} \Longrightarrow \tilde{P} = \frac{e^2\tilde{E}/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \overset{(69b)}{\Longrightarrow}$ Susceptibility: $\boxed{\tilde{\chi}_e = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}}$ ... $(69c)$

Permittivity: $\varepsilon(\omega) \equiv \tilde{\varepsilon} = 1 + \tilde{\chi}_e \overset{(69c)}{\Longrightarrow} \boxed{\tilde{\varepsilon} = 1 + \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}}$ ... $(69d)$ $\boxed{\omega \overset{\text{def}}{=} i\xi \Leftrightarrow \xi = \frac{\omega}{i} = -i\omega}$ ... $(69e)$ ... imaginary freq.

$\Longrightarrow \boxed{\varepsilon(i\xi) = 1 + \frac{e^2/m}{\omega_0^2 + \xi^2 + \gamma\xi}}$ ... $(69f)$ smoothly decaying to $\xi \to \infty$

In 3D

$(4c) \Longrightarrow E = -\partial_t\vec{A}$ $(4d) \Longrightarrow B = \vec{\nabla} \times \vec{A}$ $\epsilon = \frac{1}{2}\left(\vec{E}\vec{D} + \vec{B}\vec{H}\right) \overset{(4ab)}{=} \frac{1}{2}\left(\varepsilon_0\varepsilon\vec{E}^2 + \frac{\vec{B}}{\mu_0\mu}\frac{\varepsilon_0}{\varepsilon_0}\right) = \frac{\varepsilon_0}{2}\left(\varepsilon\vec{E}^2 + \frac{\vec{B}^2}{\mu}\frac{1}{\mu_0\varepsilon_0}\right) \overset{(6c)}{=} \frac{\varepsilon_0}{2}\left(\varepsilon\vec{E}^2 + \frac{c^2\vec{B}^2}{\mu}\right)$

$\partial_{x_1}\partial_{x_2} \to \left(\vec{\nabla}_1 \times \vec{A}\right)\cdot\left(\vec{\nabla}_2 \times \vec{A}\right) = \vec{\nabla}_1 \cdot \vec{\nabla}_2\left(\vec{A}_1 \cdot \vec{A}_2\right) - \left(\vec{\nabla}_2\vec{A}_1\right)\cdot\left(\vec{\nabla}_1\vec{A}_2\right)$ ... $(70a)$ Green's function $G \to \underline{\underline{G}} \propto \vec{A}_1 \otimes \vec{A}_2$ ... $(70b)$

$\Longrightarrow \partial_{x_1}\partial_{x_2}\tilde{G} \to \vec{\nabla}_1 \cdot \vec{\nabla}_2\operatorname{Tr}\left(\underline{\underline{\tilde{G}}}\right) - \vec{\nabla}_2 \cdot \underline{\underline{\tilde{G}}} \cdot \overleftarrow{\nabla}_1$ ... $(70c)$ with $\overleftarrow{\nabla}_1$ ... derivative applied from the right side

In index notation: $\partial_{x_1}\partial_{x_2}\tilde{G} \to \left(\partial_{x_1}\partial_{x_2} + \partial_{y_1}\partial_{y_2} + \partial_{z_1}\partial_{z_2}\right)\tilde{G}_{mm} - \partial_m^{(1)}\partial_l^{(2)}\tilde{G}_{lm}$ ... $(70d)$

Helmholtz equation: $\frac{1}{\mu}\vec{\nabla} \times \vec{\nabla} \times \underline{\underline{\tilde{G}}} - \varepsilon\frac{\omega^2}{c^2}\underline{\underline{\tilde{G}}} = \mathbb{1}\,\delta(\vec{r}_2 - \vec{r}_1)$

Example: 1D empty space Green's Function for time-independent Helmholtz equation

$\varepsilon = \mu = 1 \ldots (71a) \overset{(67)}{\Longrightarrow} \epsilon = \frac{\hbar}{2\pi}\int_0^\infty \left(\frac{\xi^2}{c^2} - \partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\, d\xi \ldots (71b)$ Helmholtz equation: $(\partial_x^2 + k^2)\,\psi(x) = \frac{1}{\varepsilon_0}q(x) \ldots (71c)$

Green's function solves $(\partial_x^2 + k^2)\,\tilde{G}(x) = \delta(x) \Longrightarrow$ Green's function solves $(\partial_x^2 + k^2)\,\tilde{G}(x) = 0$ for $x \neq 0 \Longrightarrow$

Ansatz: $\tilde{G}(x) = \begin{cases} C_- e^{ikx} + D_- e^{-ikx} & \text{for } x < 0 \\ C_+ e^{ikx} + D_+ e^{-ikx} & \text{for } x > 0 \end{cases} \ldots (71d) \Longrightarrow \tilde{G}'(x) = \begin{cases} C_- ike^{ikx} - D_- ike^{-ikx} & \text{for } x < 0 \\ C_+ ike^{ikx} - D_+ ike^{-ikx} & \text{for } x > 0 \end{cases} \ldots (71e)$

$\tilde{G}(x)$ must be continous at x=0: $\tilde{G}(0) = C_- + D_- = C_+ + D_+ \Longrightarrow$ $C_- + D_- = C_+ + D_+ \ldots (71f)$

$\tilde{G}'(x)$ must jump +1 at x=0: $\tilde{G}'(0_-) + 1 = \tilde{G}'(0_+) \overset{(71e)}{\Longrightarrow}$ $C_- ik - D_- ik + 1 = C_+ ik - D_+ ik \ldots (71g)$

Choose $\left.\begin{array}{l} C_- = 0, D_- = \frac{1}{2ik} \\ C_+ = \frac{1}{2ik}, D_+ = 0 \end{array}\right\} \begin{array}{l} \overset{(71f)}{\Longrightarrow} \frac{1}{2ik} = \frac{1}{2ik} ✓ \\ \overset{(71g)}{\Longrightarrow} -\frac{1}{2} + 1 = \frac{1}{2} ✓ \end{array}$ $\overset{(71d)}{\Longrightarrow} \tilde{G}(x) = \begin{cases} \frac{1}{2ik}e^{-ikx} & \text{for } x < 0 \\ \frac{1}{2ik}e^{ikx} & \text{for } x > 0 \end{cases} \Longrightarrow \boxed{\tilde{G} = \frac{1}{2ik}e^{ik|x|} \mathrel{\hat{=}} \frac{1}{2ik}e^{ik|x-x'|}} \ldots (72)$

Free Space 1D Energy Density $\epsilon(i\kappa)$

$\boxed{k \overset{\text{def}}{=} i\kappa \Leftrightarrow \kappa = \frac{k}{i} = -ik} \ldots (73)$ … imaginary wave number $\overset{(72)}{\Longrightarrow} \boxed{\tilde{G}(i\kappa) = -\frac{1}{2\kappa}e^{-\kappa|x|} \mathrel{\hat{=}} -\frac{1}{2\kappa}e^{-\kappa|x-x'|}} \ldots (74)$

$k = \frac{\omega}{c} \overset{(73)}{\Longrightarrow} i\kappa = \frac{\omega}{c} \overset{(69e)}{\Longrightarrow} i\kappa = \frac{i\xi}{c} \Longrightarrow \boxed{\kappa = \frac{\xi}{c} \Leftrightarrow \xi = c\kappa} \ldots (75a) \Longrightarrow \frac{d\kappa}{d\xi} = \frac{1}{c} \Longrightarrow \boxed{d\xi = c\, d\kappa} \ldots (75b) \overset{(71b)}{\Longrightarrow}$

$\epsilon = \frac{\hbar c}{2\pi}\int_0^\infty \left(\frac{\xi^2}{c^2} - \partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\, d\kappa \overset{(75a)}{=} \frac{\hbar c}{2\pi}\int_0^\infty (\kappa^2 - \partial_{x_1}\partial_{x_2})\,\tilde{G}(i\kappa)\, d\kappa \overset{(74)}{=} -\frac{\hbar c}{2\pi}\int_0^\infty (\kappa^2 - \partial_{x_1}\partial_{x_2})\left(\frac{1}{2\kappa}e^{-\kappa|x_2-x_1|}\right) d\kappa \Longrightarrow$

$\boxed{\epsilon = -\frac{\hbar c}{2\pi}\int_0^\infty (\kappa^2 + \kappa^2)\left(\frac{1}{2\kappa}e^{-\kappa|x_2-x_1|}\right) d\kappa} \,..(76)$

Example: 1D homogeneous material Green's Function for time-independent Helmholtz equation

$\overset{(67)}{\Longrightarrow} \epsilon = \frac{\hbar}{2\pi}\int_0^\infty \left(\varepsilon\frac{\xi^2}{c^2} - \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}(i\xi)\, d\xi \ldots (77a)$ Helmholtz equation: $\left(\frac{1}{\mu}\partial_x^2 + \varepsilon k^2\right)\psi(x) = \frac{1}{\varepsilon_0}q(x) \ldots (77b)$

Green's function solves $\left(\frac{1}{\mu}\partial_x^2 + \varepsilon k^2\right)\tilde{G}(x) = \delta(x) \Longrightarrow (\partial_x^2 + \varepsilon\mu k^2)\,\tilde{G}(x) = \mu\delta(x) \ldots (77c) \Longrightarrow$

Green's function solves $(\partial_x^2 + \varepsilon\mu k^2)\,\tilde{G}(x) = 0$ for $x \neq 0 \Longrightarrow$ Ansatz:

$\tilde{G}(x) = \begin{cases} C_- e^{i\sqrt{\varepsilon\mu}kx} + D_- e^{-i\sqrt{\varepsilon\mu}kx} & \ldots\ x < 0 \\ C_+ e^{i\sqrt{\varepsilon\mu}kx} + D_+ e^{-i\sqrt{\varepsilon\mu}kx} & \ldots\ x > 0 \end{cases} \ldots (77d) \Rightarrow \tilde{G}'(x) = \begin{cases} C_- i\sqrt{\varepsilon\mu}ke^{i\sqrt{\varepsilon\mu}kx} - D_- i\sqrt{\varepsilon\mu}ke^{-i\sqrt{\varepsilon\mu}kx} & \ldots\ x < 0 \\ C_+ i\sqrt{\varepsilon\mu}ke^{i\sqrt{\varepsilon\mu}kx} - D_+ i\sqrt{\varepsilon\mu}ke^{-i\sqrt{\varepsilon\mu}kx} & \ldots\ x > 0 \end{cases} \ldots (77e)$

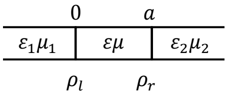
$\tilde{G}(x)$ must be continous at x=0: $\tilde{G}(0) = C_- + D_- = C_+ + D_+ \Longrightarrow$ $C_- + D_- = C_+ + D_+ \ldots (77f)$

$\tilde{G}'(x)$ must jump $+\mu$ at x=0: $\tilde{G}'(0_-) + \mu = \tilde{G}'(0_+) \overset{(77e)}{\Longrightarrow}$ $C_- i\sqrt{\varepsilon\mu}k - D_- i\sqrt{\varepsilon\mu}k + \mu = C_+ i\sqrt{\varepsilon\mu}k - D_+ i\sqrt{\varepsilon\mu}k \ldots (77g)$

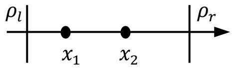
Choose $\left.\begin{array}{l} C_- = 0, D_- = \frac{\mu}{2i\sqrt{\varepsilon\mu}k} \\ C_+ = \frac{\mu}{2i\sqrt{\varepsilon\mu}k}, D_+ = 0 \end{array}\right\} \begin{array}{l} \overset{(77f)}{\Longrightarrow} \frac{\mu}{2i\sqrt{\varepsilon\mu}k} = \frac{\mu}{2i\sqrt{\varepsilon\mu}k} ✓ \\ \overset{(77g)}{\Longrightarrow} -\frac{\mu}{2} + \mu = \frac{\mu}{2} ✓ \end{array}$ $\overset{(77d)}{\Longrightarrow} \tilde{G}(x) = \begin{cases} \frac{\mu}{2i\sqrt{\varepsilon\mu}k}e^{-i\sqrt{\varepsilon\mu}kx} & \text{for } x < 0 \\ \frac{\mu}{2i\sqrt{\varepsilon\mu}k}e^{i\sqrt{\varepsilon\mu}kx} & \text{for } x > 0 \end{cases} \Longrightarrow$

$\tilde{G} = \frac{\mu}{2i\sqrt{\varepsilon\mu}k}e^{i\sqrt{\varepsilon\mu}k|x|} \overset{(73)}{=} -\frac{\mu}{2\sqrt{\varepsilon\mu}\kappa}e^{-\sqrt{\varepsilon\mu}\kappa|x|} \Rightarrow \boxed{\tilde{G}_0 = -\frac{\mu}{2w}e^{-w|x|} \mathrel{\hat{=}} -\frac{\mu}{2w}e^{-w|x_2-x_1|}} \ldots (78)$ with $\boxed{w \overset{\text{def}}{=} \sqrt{\varepsilon\mu}\kappa \overset{(73)}{=} -i\sqrt{\varepsilon\mu}k} \ldots (79)$

Example: Slab between two semi-infinite half-spaces



We put x_1 and x_2 inside the region $0 < x < a$.
We consider $x_2 > x_1$ with x_1… "source" (fix)
and x_2… free variable ("observation target")
ρ_l, ρ_r … Fresnel Reflection Coefficients left and right



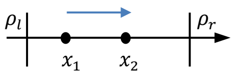
source target

Infinite # of round-trips:

$P_{\text{inf}} = \underbrace{1}_{no\ roundtrip} + \underbrace{\rho_l\rho_r e^{-2aw}}_{1\ roundtrip} + \underbrace{(\rho_l\rho_r e^{-2aw})^2}_{2\ roundtrips} + \cdots = \sum_{m=0}^{\infty}(\rho_l\rho_r e^{-2aw})^m \Longrightarrow \boxed{P_{\text{inf}} = \frac{1}{1-\rho_l\rho_r e^{-2aw}}} \ldots (80)$

Trying to express all possible paths from x_1 to x_2 in Green's function $\tilde{G}_1$

contributions 1

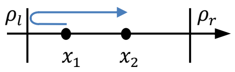


Direct propagation $x_1 \to x_2$: $e^{i\sqrt{\varepsilon\mu}k(x_2-x_1)} \overset{(79)}{=} e^{-w(x_2-x_1)}$

1 extra roundtrip: $e^{-w(x_2-x_1)}\rho_l\rho_r e^{-2aw}$, 2 extra roundtrips: $e^{-w(x_2-x_1)}(\rho_l\rho_r e^{-2aw})^2$, … $\Longrightarrow$

$P_{\text{C1}} = e^{-w(x_2-x_1)}P_{\text{inf}} \overset{(80)}{=} \frac{e^{-w(x_2-x_1)}}{1-\rho_l\rho_r e^{-2aw}} \ldots (81a)$

contributions 2

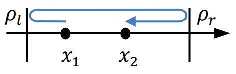


Propagation $x_1 \to$ L $\to x_2$: $e^{i\sqrt{\varepsilon\mu}kx_1}\rho_l e^{i\sqrt{\varepsilon\mu}kx_2} = \rho_l e^{i\sqrt{\varepsilon\mu}k(x_1+x_2)} \overset{(79)}{=} \rho_l e^{-w(x_1+x_2)}$

1 extra roundtrip: $\rho_l e^{-w(x_1+x_2)}\rho_l\rho_r e^{-2aw}$, 2 extra roundtrips: $\rho_l e^{-w(x_1+x_2)}(\rho_l\rho_r e^{-2aw})^2$, … $\Rightarrow$

$P_{\text{C2}} = e^{-w(x_1+x_2)}P_{\text{inf}} \overset{(80)}{=} \frac{\rho_l e^{-w(x_1+x_2)}}{1-\rho_l\rho_r e^{-2aw}} \ldots (81b)$

contributions 3



Prop. $x_1 \to$ L $\to$ R$\to x_2$: $\rho_l\rho_r e^{i\sqrt{\varepsilon\mu}k(x_1+a+(a-x_2))} \overset{(79)}{=} \rho_l\rho_r e^{-w(2a+x_1-x_2)} = e^{-w(x_1-x_2)}\rho_l\rho_r e^{-2aw}$

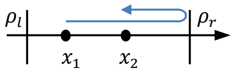
1 extra roundtrip: $e^{-w(x_1-x_2)}(\rho_l\rho_r e^{-2aw})^2$, 2 extra roundtrips: $e^{-w(x_1-x_2)}(\rho_l\rho_r e^{-2aw})^3$, … $\Longrightarrow$

We want: $e^{-w(x_1-x_2)}\rho_l\rho_r e^{-2aw} + e^{-w(x_1-x_2)}(\rho_l\rho_r e^{-2aw})^2 + \cdots$

Notice: $e^{-w(x_1-x_2)}P_{\text{inf}} = \underbrace{e^{-w(x_1-x_2)}}_{\text{excess term}} + e^{-w(x_1-x_2)}\rho_l\rho_r e^{-2aw} + e^{-w(x_1-x_2)}(\rho_l\rho_r e^{-2aw})^2 + \cdots \Rightarrow$

$P_{\text{C3}} = e^{-w(x_1-x_2)}P_{\text{inf}} = \frac{e^{-w(x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}} \ldots (81c)$, but we must compensate excess term $e^{-w(x_1-x_2)}$!

contributions 4



Propagation $x_1 \to$ R $\to x_2$: $\rho_r e^{i\sqrt{\varepsilon\mu}k((a-x_1)+(a-x_2))} = \rho_r e^{i\sqrt{\varepsilon\mu}k(2a-x_1-x_2)} \overset{(79)}{=} \rho_r e^{-w(2a-x_1-x_2)}$

1 extra roundtrip: $\rho_r e^{-w(2a-x_1-x_2)}\rho_l\rho_r e^{-2aw}$, 2 extra: $\rho_r e^{-w(2a-x_1-x_2)}(\rho_l\rho_r e^{-2aw})^2$, … $\Longrightarrow$

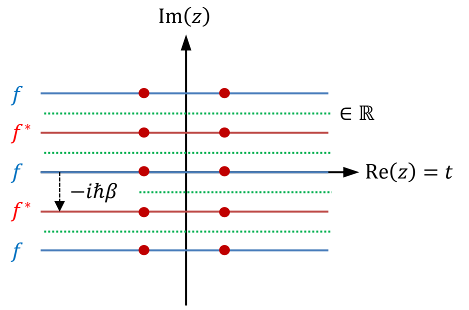
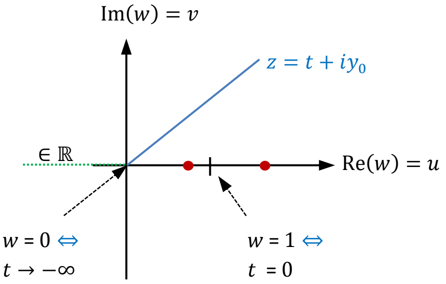
$P_{\text{C4}} = e^{-w(2a-x_1-x_2)}P_{\text{inf}} \overset{(80)}{=} \frac{\rho_r e^{-w(2a-x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}} \ldots (81d)$

creating $\tilde{G}_1$ analogously to (78)

$\tilde{G}_1 = -\frac{\mu}{2w}(P_{\text{C1}} + P_{\text{C2}} + P_{\text{C3}} + P_{\text{C3}}) \overset{(81abcd)}{\Longrightarrow} \boxed{\tilde{G}_1 = -\frac{\mu}{2w}\frac{e^{-w(x_2-x_1)}+\rho_l e^{-w(x_1+x_2)}+e^{-w(x_1-x_2)}+\rho_r e^{-w(2a-x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}}} \ldots (82)$

Compensation for excess term. Full Green's function.	If $x_2 > x_1$: P_{C3} contains the excess term $-\frac{\mu}{2w}e^{-w(x_1-x_2)}$, to be compensated with $\tilde{G}_{\text{comp}} = +\frac{\mu}{2w}e^{-w(x_1-x_2)}$ … (83a) If $x_2 < x_2$: P_{C1} contains the excess term $-\frac{\mu}{2w}e^{-w(x_2-x_1)}$, to be compensated with $\tilde{G}_{\text{comp}} = +\frac{\mu}{2w}e^{-w(x_2-x_1)}$ … (83b) We define: $\boxed{\tilde{G}_2 \overset{\text{def}}{=} \frac{\mu}{2w}\left(e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)}$ … (84) Then, $\left.\begin{array}{l}\text{if } x_2 > x_1\text{: } \tilde{G}_0+\tilde{G}_2 \overset{(80)}{=} -\frac{\mu}{2w}e^{-w(x_2-x_1)}+\tilde{G}_2 \overset{(84)}{=} \frac{\mu}{2w}\left(\cancel{-e^{-w(x_2-x_1)}}+e^{-w(x_1-x_2)}+\cancel{e^{-w(x_2-x_1)}}\right) \overset{(83a)}{=} \tilde{G}_{\text{comp}} \\ \text{if } x_2 < x_1\text{: } \tilde{G}_0+\tilde{G}_2 \overset{(80)}{=} -\frac{\mu}{2w}e^{-w(x_1-x_2)}+\tilde{G}_2 \overset{(84)}{=} \frac{\mu}{2w}\left(\cancel{-e^{-w(x_1-x_2)}}+\cancel{e^{-w(x_1-x_2)}}+e^{-w(x_2-x_1)}\right) \overset{(83b)}{=} \tilde{G}_{\text{comp}}\end{array}\right\}$ … (85) $\Longrightarrow$ $\tilde{G} = \tilde{G}_1+\tilde{G}_{\text{comp}} \overset{(85)}{\Longrightarrow} \tilde{G} = \tilde{G}_1+(\tilde{G}_0+\tilde{G}_2) \Longrightarrow \boxed{\tilde{G} = \tilde{G}_0+\tilde{G}_1+\tilde{G}_2}$ … (86)
renormalization	Every considered particle is driven by vacuum fluctuations and emits (virtual) em waves. On average, the emissions of different particles are incoherent. Therefore, each particle interacts on average only with the waves emitted by itself. $\tilde{G}_1$ contains the direct propagation between x_1 and x_2 in either direction (as expressed in $\tilde{G}_0$). When we consider the limit $x_1 \to x_2$, this leads to a diverging energy density. But this direct self-interaction is already contained in the formation of the particle and should not be included into the calculation. Therefore, we need subtract the self-interaction expressed by $\tilde{G}_0$: $\tilde{G}_{\text{ren}} = \tilde{G}-\tilde{G}_0 \overset{(86)}{\Longrightarrow} \boxed{\tilde{G}_{\text{ren}} = \tilde{G}_1+\tilde{G}_2}$ … (87a) $\overset{(82)(84)}{\Longrightarrow}$ $\boxed{\tilde{G}_{\text{ren}} = -\frac{\mu}{2w}\frac{e^{-w(x_2-x_1)}+\rho_l e^{-w(x_1+x_2)}+e^{-w(x_1-x_2)}+\rho_r e^{-w(2a-x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}}+\frac{\mu}{2w}\left(e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)}$ … (87b)
Energy density: Lifshitz Formula	(67) $\Longrightarrow \epsilon = \frac{\hbar}{2\pi}\int_0^\infty\left(\varepsilon\frac{\xi^2}{c^2}-\frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}_{\text{ren}}\,d\xi\Big\vert_{x_1\to x_2} \overset{(87a)}{=} \frac{\hbar}{2\pi}\int_0^\infty\left(\varepsilon\frac{\xi^2}{c^2}-\frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}_{\text{ren}}\,d\xi\Big\vert_{x_1\to x_2} \overset{(75b)}{\Longrightarrow}$ $\epsilon = \frac{\hbar c}{2\pi}\int_0^\infty\left(\varepsilon\frac{\xi^2}{c^2}-\frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}_{\text{ren}}\,d\kappa\Big\vert_{x_1\to x_2} \overset{(75a)}{\Longrightarrow} \boxed{\epsilon = \frac{\hbar c}{2\pi}\int_0^\infty\left(\varepsilon\kappa^2-\frac{1}{\mu}\partial_{x_1}\partial_{x_2}\right)\tilde{G}_{\text{ren}}\,d\kappa\Big\vert_{x_1\to x_2}}$ … (88) $\frac{1}{\mu}\partial_{x_1}\partial_{x_2}\tilde{G}_{\text{ren}} \overset{(87b)}{=} \frac{1}{\mu}\partial_{x_1}\partial_{x_2}\left(-\frac{e^{-w(x_2-x_1)}+\rho_l e^{-w(x_1+x_2)}+e^{-w(x_1-x_2)}+\rho_r e^{-2aw}e^{w(x_1+x_2)}}{1-\rho_l\rho_r e^{-2aw}}+e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)\frac{\mu}{2w}$ $= \frac{1}{\mu}\left(-\frac{(-w^2)e^{-w(x_2-x_1)}+(w^2)\rho_l e^{-w(x_1+x_2)}+(-w^2)e^{-w(x_1-x_2)}+(w^2)\rho_r e^{-2aw}e^{w(x_1+x_2)}}{1-\rho_l\rho_r e^{-2aw}}+(-w^2)e^{-w(x_1-x_2)}+(-w^2)e^{-w(x_2-x_1)}\right)\frac{\mu}{2w}$ $= \frac{1}{\mu}(-w^2)\left(-\frac{e^{-w(x_2-x_1)}+e^{-w(x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}}+e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)\frac{\mu}{2w}+\frac{1}{\mu}w^2\left(-\frac{\rho_l e^{-w(x_1+x_2)}+\rho_r e^{-2aw}e^{w(x_1+x_2)}}{1-\rho_l\rho_r e^{-2aw}}\right)\frac{\mu}{2w} \overset{(79)}{\Longrightarrow}$ $= -\varepsilon\kappa^2\underbrace{\left(-\frac{e^{-w(x_2-x_1)}+e^{-w(x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}}+e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)\frac{\mu}{2w}}_{\tilde{G}_{\text{A}}}+\varepsilon\kappa^2\underbrace{\left(-\frac{\rho_l e^{-w(x_1+x_2)}+\rho_r e^{-2aw}e^{w(x_1+x_2)}}{1-\rho_l\rho_r e^{-2aw}}\right)\frac{\mu}{2w}}_{\tilde{G}_{\text{B}}} \overset{(88)}{\Longrightarrow}$ $\epsilon = \frac{\hbar c}{2\pi}\int_0^\infty\left((\varepsilon\kappa^2+\varepsilon\kappa^2)\tilde{G}_{\text{A}}+\cancel{(\varepsilon\kappa^2-\varepsilon\kappa^2)\tilde{G}_{\text{B}}}\right)d\kappa\Big\vert_{x_1\to x_2} \Longrightarrow$ $\epsilon = \frac{\hbar c}{2\pi}\int_0^\infty\left(2\varepsilon\kappa^2\left(-\frac{e^{-w(x_2-x_1)}+e^{-w(x_1-x_2)}}{1-\rho_l\rho_r e^{-2aw}}+e^{-w(x_1-x_2)}+e^{-w(x_2-x_1)}\right)\frac{\mu}{2w}\right)d\kappa\Big\vert_{x_1\to x_2=x} \Longrightarrow$ $\epsilon = \frac{\hbar c}{2\pi}\frac{\mu}{w}\varepsilon\kappa^2\int_0^\infty\left(-\frac{1+1}{1-\rho_l\rho_r e^{-2aw}}+1+1\right)d\kappa = -\frac{\hbar c}{\pi}\frac{\mu}{w}\varepsilon\kappa^2\int_0^\infty\left(\frac{1}{1-\rho_l\rho_r e^{-2aw}}-1\right)d\kappa = -\frac{\hbar c}{\pi}\frac{\mu}{w}\varepsilon\kappa^2\int_0^\infty\frac{\cancel{1-1}+\rho_l\rho_r e^{-2aw}}{1-\rho_l\rho_r e^{-2aw}}d\kappa \Longrightarrow$ $\epsilon = -\frac{\hbar c}{\pi}\frac{\mu}{w}\varepsilon\kappa^2\int_0^\infty\frac{\rho_l\rho_r e^{-2aw}}{1-\rho_l\rho_r e^{-2aw}}\cdot\frac{(\rho_l\rho_r)^{-1}e^{2aw}}{(\rho_l\rho_r)^{-1}e^{2aw}}d\kappa = -\frac{\hbar c}{\pi}\frac{1}{w}\varepsilon\mu\kappa^2\int_0^\infty\frac{1}{(\rho_l\rho_r)^{-1}e^{2aw}-1}d\kappa \overset{(87a)}{=} -\frac{\hbar c}{\pi}\frac{1}{w}w^2\int_0^\infty\frac{1}{(\rho_l\rho_r)^{-1}e^{2aw}-1}d\kappa \Longrightarrow$ $\boxed{\epsilon = -\frac{\hbar c}{\pi}\int_0^\infty\frac{w}{(\rho_l\rho_r)^{-1}e^{2aw}-1}d\kappa}$ … (89) … Lifshitz Formula. Converges rapidly for $w \gg \frac{1}{a}$. Characteristic imaginary wavenumber: $\kappa \leq \frac{1}{a}$ Energy density "inside" (between 0 and a): $\epsilon(x) \propto \Theta(x)\,\Theta(a-x)$ Force density: $f(x) = -\partial_a\,\epsilon(x) \propto \delta(x)-\delta(a-x) = \partial_x\,\sigma(x)$ with stress density: $\sigma(x) \propto \Theta(x)\,\Theta(a-x)$
Default Casimir Effect Formula	$\varepsilon = \mu = 1; \rho_l = \rho_r = -1 \overset{(89)}{\Longrightarrow} \epsilon = \overset{?}{-}\frac{\hbar c}{\pi}\int_0^\infty\frac{w}{e^{2aw}-1}d\kappa \overset{(79)}{=} \overset{?}{-}\frac{\hbar c}{\pi}\int_0^\infty\frac{\kappa}{e^{2a\kappa}-1}d\kappa \Longrightarrow \boxed{\epsilon = \overset{?}{-}\frac{\hbar c}{24}\frac{\pi}{a^2}}$ … (90) ✓cf. (45)
In 3D	Planar mirrors, piece-wise homogeneous medium. We must take polarizations E and H into account (which are well-separated in the planar case). Fourier-transform in xy-direction: $xy \to uv$ $\boxed{\epsilon = \overset{?}{-}\frac{\hbar c}{\pi}\int_0^\infty\underbrace{\frac{1}{2\pi^2}\iint_{-\infty}^{\infty}\left(\frac{w}{A_E}+\frac{w}{A_H}\right)du\,dv}_{\text{inverse FT}}\,d\kappa\ \ \text{with}\ \ w = \sqrt{u^2+v^2+\varepsilon\mu\kappa^2}\ \ \text{and}\ \ \underbrace{A = (\rho_l\rho_r)^{-1}e^{2wa}-1}_{\text{separate for 2 polarizations}}}$ … (91) Perfect reflectors: E (electric) polarization: $\rho_l = \rho_r = -1$, H (magnetic) polarization: $\rho_l = \rho_r = 1$ Use spherical coordinates in (u, v, w) (radius w) $\kappa \geq 0 \Longrightarrow$half space $\theta \in \left[0, \frac{\pi}{2}\right] \Longrightarrow$ $\epsilon = -\frac{\hbar c}{\pi}\underbrace{2}_{\text{polari–zations}}\frac{1}{2\pi^2}\underbrace{2\pi}_{\int_0^{2\pi}d\varphi}\int_0^\infty\int_0^{\pi/2}\frac{w^3\sin(\vartheta)}{e^{2aw}-1}d\vartheta\,dw = -\frac{\hbar c}{\pi^2}\int_0^\infty\frac{w^3}{e^{2aw}-1}dw \Longrightarrow \boxed{\epsilon = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}}$ … (91) ✓see (49)
electric and magnetic mirror	Electric mirror (typical): $E_\perp = 0$ Reflection coefficient E-Field: $\rho_E = -1$ Reflection coefficient H-Field: $\rho_H = +1$ Magnetic mirror (difficult): $H_\perp = 0$ Reflection coefficient E-Field: $\rho_E = +1$ Reflection coefficient H-Field: $\rho_H = -1$ Assuming electric and magnetic mirror facing each other: $\varepsilon = \mu = 1; \rho_l = -1, \rho_r = 1 \overset{(89)}{\Longrightarrow} \epsilon = -\frac{\hbar c}{\pi}\int_0^\infty\frac{w}{-e^{2aw}-1}d\kappa \overset{(79)}{=} \frac{\hbar c}{\pi}\int_0^\infty\frac{\kappa}{e^{2a\kappa}+1}d\kappa \Longrightarrow \boxed{\epsilon = \frac{7}{8}\frac{\hbar c}{24}\frac{\pi}{a^2}}$ … (92) repulsive!
$\varepsilon_2 > \varepsilon > \varepsilon_1$	An ε-hierarchy ($\varepsilon_2 > \varepsilon > \varepsilon_1$) is more realistic for a broad range of frequencies. $\Longrightarrow$ ρ_l and ρ_r have opposite signs $\Longrightarrow \rho_l\rho_r < 0$. Has been done experimentally with Silicon and gold (Munday et al, Nature 2009). Also: Zhao et al, Science 2019.

2.3 Temperature

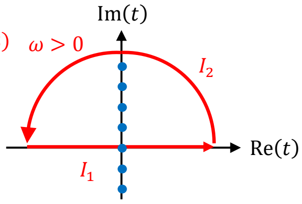
therm. wavelength	Thermal Wavelength: $\boxed{\lambda_{th}=\frac{c}{f_{th}}}$ ... $(93a)$ $\lambda_{th}\approx 50\mu\text{m}$ at room temperature. Thermal Frequency: $\boxed{f_{th}=\frac{k_BT}{h}}$ ... $(93b)$ Thermal radiation relevant when typical distances $\geq\frac{1}{4\pi}\lambda_{th}$ (with λ_{th} being the peak of the Planck spectrum)
Vacuum vs. thermal states	We could derive the correlation function $(59c)$ only by assuming that Γ decays zo zero in (complex) infinity. This is not the case for thermal states.
Thermal state for an arbitrary operator	$\boxed{\langle\hat{F}\rangle=\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{F}\right)}$ ... $(94a)$ with $\boxed{\beta\overset{\text{def}}{=}\frac{1}{k_BT}}$ ... $(94b)$ and Partition Function $\boxed{Z=\mathrm{Tr}\left(e^{-\beta\hat{H}}\right)}$ ... $(94c)$ for normalization Thermal state expectation value $\boxed{f(t)=\frac{\varepsilon_0c}{\hbar}\langle\hat{A}_2\hat{A}_1\rangle \text{ with } t=t_2-t_1}$ ... $(94d)$ $\overset{(94a)}{\Longrightarrow}$ $\boxed{f(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{A}_2\hat{A}_1\right)}$ ... $(94e)$
Kubo-Martin-Schwinger-Relation	$\Longrightarrow f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{A}_2\hat{A}_1\right)^*=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\left(\hat{A}_2\hat{A}_1\right)^\dagger e^{-\beta\hat{H}^\dagger}\right)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\hat{A}_1^\dagger\hat{A}_2^\dagger e^{-\beta\hat{H}^\dagger}\right)\Big\vert\ \hat{A}_1=\hat{A}_1^\dagger,\hat{A}_2=\hat{A}_2^\dagger,\hat{H}=\hat{H}^\dagger\Longrightarrow$ $f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\hat{A}_1\hat{A}_2e^{-\beta\hat{H}}\right)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\hat{A}_1\hat{A}_2e^{-\frac{\beta}{2}\hat{H}}e^{-\frac{\beta}{2}\hat{H}}\right)\Big\vert\ \mathrm{Tr}(ABCD)=\mathrm{Tr}(DABA)\Longrightarrow$ $f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(e^{-\frac{\beta}{2}\hat{H}}\hat{A}_1\hat{A}_2e^{-\frac{\beta}{2}\hat{H}}\right)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(e^{-\frac{\beta}{2}\hat{H}}\hat{A}_1e^{\frac{\beta}{2}\hat{H}}e^{-\beta\hat{H}}e^{\frac{\beta}{2}\hat{H}}\hat{A}_2e^{-\frac{\beta}{2}\hat{H}}\right)\Big\vert -\frac{\beta}{2}\hat{H}=\frac{i}{\hbar}\left(\frac{i\hbar\beta}{2}\right)\hat{H},\frac{\beta}{2}\hat{H}=\frac{i}{\hbar}\left(-\frac{i\hbar\beta}{2}\right)\hat{H}\Longrightarrow$ $f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\underbrace{e^{\frac{i}{\hbar}\left(\frac{i\hbar\beta}{2}\right)\hat{H}}\hat{A}_1e^{\frac{i}{\hbar}\left(-\frac{i\hbar\beta}{2}\right)\hat{H}}}_{\substack{\text{time propagation with}\\\text{imaginary time }+\frac{i\hbar\beta}{2}}}e^{-\beta\hat{H}}\underbrace{e^{\frac{i}{\hbar}\left(-\frac{i\hbar\beta}{2}\right)\hat{H}}\hat{A}_2e^{\frac{i}{\hbar}\left(\frac{i\hbar\beta}{2}\right)\hat{H}}}_{\substack{\text{time propagation with}\\\text{imaginary time }-\frac{i\hbar\beta}{2}}}\right)\Longrightarrow f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(\hat{A}_1\left(t_1+\frac{i\hbar\beta}{2}\right)e^{-\beta\hat{H}}\hat{A}_2\left(t_2-\frac{i\hbar\beta}{2}\right)\right)\Longrightarrow$ $f^*(t)=\frac{\varepsilon_0c}{\hbar}\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{A}_2\left(t_2-\frac{i\hbar\beta}{2}\right)\hat{A}_1\left(t_1+\frac{i\hbar\beta}{2}\right)\right)\overset{(94e)}{\Longrightarrow}\boxed{f^*(t)=f(t_*)\text{ with }t_*=t_2-\frac{i\hbar\beta}{2}-\left(t_1+\frac{i\hbar\beta}{2}\right)=t-i\hbar\beta}$ ... (95)
Conformal Transformation	For real times t, the function f has poles (δ-peaks) left and right of zero. The analytic continuation of f on the line $t_*=t-i\hbar\beta$ yields the complex conjugate f^* with also two poles. Actually, the analytic continuation of f is periodic in the positive and negative imaginary direction (a ladder of lines parallel to the real axis with alternating values f and f^*, each of which has poles). Because of the Schwarz reflection principle, the analytical continuation is real-valued between all f and f^* (green dotted lines). The function decays neither in the upper nor in the lower complex half-plane. The Kramers-Kronig relation cannot be applied. We introduce the following conformal transformation which maps the periodic structure to an infinite stack of Riemann sheets. 
	$\boxed{w=e^{\frac{2\pi}{\hbar\beta}z}\text{ with }z\text{ ... complex time}}$ ... $(96a)$ so that $\boxed{\mathrm{Re}(z)=t}$ ... $(96b)$ With this map, each horizontal strip between two neighboring lines f and f^* gets mapped on a separate Riemann sheet: $w(z\pm in\hbar\beta)\overset{(96a)}{=}e^{\frac{2\pi}{\hbar\beta}(z\pm in\hbar\beta)}=e^{\frac{2\pi}{\hbar\beta}z}e^{2\pi ni}=e^{\frac{2\pi}{\hbar\beta}z}\overset{(96a)}{\Longrightarrow}$ $\boxed{w(z\pm i\hbar\beta n)=w(z)\text{ with }n\in\mathbb{N}}$ ... $(96c)$ Horizontal lines in the z-plane get mapped on radial lines in the w-plane. $\mathrm{Re}(z)$ determines the radial distance, $\mathrm{Im}(z)$ the polar angle: $e^z=\underbrace{e^{\mathrm{Re}(z)}}_{\text{radial distance}}\underbrace{(\cos(\mathrm{Im}(z))+i\,\mathrm{sincos}(\mathrm{Im}(z)))}_{\text{polar angle}}$ In particular, t=0 gets mapped on w=1, t→ $-\infty$ gets mapped on w=0. Real times t are mapped on $\mathrm{Re}(w)=u$ $\boxed{t\in[-\infty,\infty]\Leftrightarrow u\in[0,\infty]}$ ... $(96d)$ The real valued dotted green horizontal lines are mapped to $\mathrm{Re}(w)<0\Leftrightarrow u<0$ ... $(96e)$ 
	Now we can use the Kramers-Kronig relation in the w-plane. Kramers Kronig in general: $\mathrm{Re}(f)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(\xi'))}{\xi'-\xi}d\xi'$ $\xi\to u,f\to f\circ z,\ f(\xi')\to f(z(u'))\Longrightarrow\mathrm{Re}(f)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(z(u')))}{u'-u}du'\overset{(96e)}{\Longrightarrow}\mathrm{Re}(f)=\frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Im}(f(z(u')))}{u'-u}du'$ ... $(96f)$ $u=\mathrm{Re}(w)\overset{(96a)}{\Longrightarrow}u=e^{\frac{2\pi}{\hbar\beta}\mathrm{Re}(z)}\overset{(96b)}{\Longrightarrow}u=e^{\frac{2\pi}{\hbar\beta}t}$ ... $(96g)$ $\Longrightarrow\frac{du'}{dt'}=\frac{2\pi}{\hbar\beta}e^{\frac{2\pi}{\hbar\beta}t'}\Longrightarrow du'=\frac{2\pi}{\hbar\beta}e^{\frac{2\pi}{\hbar\beta}t'}dt'=\frac{2\pi}{\hbar\beta}\frac{1}{e^{-\frac{2\pi}{\hbar\beta}t'}}dt'\overset{(96f)}{\Longrightarrow}$ $\mathrm{Re}(f)=\frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(z'))}{u'-u}\frac{1}{e^{-\frac{2\pi}{\hbar\beta}t'}}dt'\overset{(96b)}{=}\frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(t'))}{u'-u}\frac{1}{e^{-\frac{2\pi}{\hbar\beta}t'}}dt'\overset{(96g)}{=}\frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(t'))}{e^{\frac{2\pi}{\hbar\beta}t'}-e^{\frac{2\pi}{\hbar\beta}t}}\frac{1}{e^{-\frac{2\pi}{\hbar\beta}t'}}dt'\Longrightarrow$ $\mathrm{Re}(f)=\frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(t'))}{1-e^{\frac{2\pi}{\hbar\beta}(t-t')}}dt'\overset{(59a)}{\Longrightarrow}\boxed{K_{th}=\frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}(f(t'))}{e^{\frac{2\pi}{\hbar\beta}(t-t')}-1}dt'}$ ... (97)

Calculate $\tilde{f}_1(\omega)$ with convolution theorem and residue theorem

Sign?

$$(97) \Longrightarrow K_{th} = \frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{e^{\frac{2\pi}{\hbar\beta}(t-t')}-1}\mathrm{Im}(f(t'))\,dt' = \frac{2}{\hbar\beta}\mathcal{P}\int_{-\infty}^{\infty}f_1(t-t')\,f_2(t')\,dt' \xRightarrow{(60c)}$$

$$\boxed{K_{th} = \frac{1}{2\pi}\frac{2}{\hbar\beta}\int_{-\infty}^{\infty}\tilde{f}_1(\omega)\,\tilde{f}_2(\omega)\,e^{-i\omega t}\,d\omega \text{ with } f_1(t) = \frac{1}{e^{\frac{2\pi}{\hbar\beta}t}-1} \text{ and } f_2(t) = \mathrm{Im}(f(t))} \ldots (98)$$



$$\tilde{f}_1(\omega) \overset{(90a)}{=} \int_{-\infty}^{\infty}\frac{1}{e^{\frac{2\pi}{\hbar\beta}t}-1}e^{i\omega t}\,dt \Longrightarrow \boxed{\tilde{f}_1(\omega) = \int_{-\infty}^{\infty}\frac{e^{i\omega t}}{e^{\alpha t}-1}dt \overset{\text{def}}{=} I_1 \text{ with } \alpha = \frac{2\pi}{\hbar\beta}} \ldots (99a)$$

We consider $\omega > 0 \Longrightarrow \omega \to i\infty \Longrightarrow e^{i\omega t}\big|_{\omega\to i\infty} \to 0$ ✓ $\Longrightarrow$ close over the upper plane

$\frac{e^{i\omega t}}{e^{\alpha t}-1}$ has poles at $e^{\alpha t} - 1 = 0 \Longrightarrow e^{\alpha t} = 1 \Longrightarrow \alpha t = 2\pi i m \Longrightarrow \boxed{t = \frac{2\pi i m}{\alpha}, m \in \mathbb{N}} \ldots (99b)$

Residue Theorem (note that we take only half the value for the residue at $t = 0$, because we integrate directly through it):

$$I_1 + \underbrace{\cancel{I_2}}_{0} = \pi i\,\mathrm{Res}_{t\to 0}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) + 2\pi i\sum_{m=1}^{\infty}\mathrm{Res}_{t\to\frac{2\pi i m}{\alpha}}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) \ldots (99c)$$

$$\mathrm{Res}_{t\to 0}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) = \lim_{t\to 0} t\frac{e^{i\omega t}}{e^{\alpha t}-1} = \lim_{t\to 0}\frac{\partial_t(te^{i\omega t})}{\partial_t(e^{\alpha t}-1)} = \lim_{t\to 0}\frac{e^{i\omega t}+i\omega e^{i\omega t}}{\alpha e^{\alpha t}} = \frac{1+0}{\alpha} \Longrightarrow \boxed{\mathrm{Res}_{t\to 0}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) = \frac{1}{\alpha}} \xRightarrow{(99c)}$$

$$\underline{I_1 = \frac{\pi i}{\alpha} + 2\pi i\sum_{m=1}^{\infty}\mathrm{Res}_{t\to\frac{2\pi i m}{\alpha}}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right)} \ldots (99d) \quad \mathrm{Res}_{t\to\frac{2\pi i m}{\alpha}}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) = \lim_{t\to\frac{2\pi i m}{\alpha}}\left(t - \frac{2\pi i m}{\alpha}\right)\frac{e^{i\omega t}}{e^{\alpha t}-1} = \lim_{t\to\frac{2\pi i m}{\alpha}}\frac{\partial_t\left(te^{i\omega t}-\frac{2\pi i m}{\alpha}e^{i\omega t}\right)}{\partial_t(e^{\alpha t}-1)}$$

$$\mathrm{Res}_{t\to\frac{2\pi i m}{\alpha}}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) = \lim_{t\to\frac{2\pi i m}{\alpha}}\frac{e^{i\omega t}+i\omega te^{i\omega t}+\omega\frac{2\pi m}{\alpha}e^{i\omega t}}{\alpha e^{\alpha t}} = \frac{e^{-\omega\frac{2\pi m}{\alpha}}\cancel{-\omega\frac{2\pi m}{\alpha}e^{-\omega\frac{2\pi m}{\alpha}}+\omega\frac{2\pi m}{\alpha}e^{-\omega\frac{2\pi m}{\alpha}}}}{\alpha\cancel{e^{2\pi i m}}} \Rightarrow \boxed{\mathrm{Res}_{t\to\frac{2\pi i m}{\alpha}}\left(\frac{e^{i\omega t}}{e^{\alpha t}-1}\right) = \frac{1}{\alpha}e^{-\omega\frac{2\pi m}{\alpha}}} \xRightarrow{(99d)}$$

$$I_1 = \frac{\pi i}{\alpha} + \frac{2\pi i}{\alpha}\sum_{m=1}^{\infty}e^{-\frac{2\pi\omega}{\alpha}m} = \frac{2\pi i}{\alpha}\left(\frac{1}{2} + \sum_{m=1}^{\infty}e^{-\frac{2\pi\omega}{\alpha}m}\right) = \frac{2\pi i}{\alpha}\left(\frac{1}{2} + \sum_{m=0}^{\infty}e^{-\frac{2\pi\omega}{\alpha}m} - 1\right) = \frac{2\pi i}{\alpha}\left(-\frac{1}{2} + \sum_{m=0}^{\infty}\left(e^{-\frac{2\pi\omega}{\alpha}}\right)^m\right) \Longrightarrow$$

$$I_1 = \frac{2\pi i}{\alpha}\left(-\frac{1}{2} + \frac{1}{1-e^{-\frac{2\pi\omega}{\alpha}}}\right) = \frac{2\pi i}{\alpha}\left(-\frac{1-e^{-\frac{2\pi\omega}{\alpha}}}{2-2e^{-\frac{2\pi\omega}{\alpha}}} + \frac{2}{2-2e^{-\frac{2\pi\omega}{\alpha}}}\right) = \frac{i\pi}{\alpha}\left(\frac{e^{-\frac{2\pi\omega}{\alpha}}-1}{1-1e^{-\frac{2\pi\omega}{\alpha}}} + \frac{2}{1-1e^{-\frac{2\pi\omega}{\alpha}}}\right) = \frac{i\pi}{\alpha}\frac{e^{-\frac{2\pi\omega}{\alpha}}+1}{1-1e^{-\frac{2\pi\omega}{\alpha}}}\cdot\frac{e^{\frac{\pi\omega}{\alpha}}}{e^{\frac{\pi\omega}{\alpha}}} \Longrightarrow$$

$$I_1 = \frac{i\pi}{\alpha}\frac{e^{-\frac{\pi\omega}{\alpha}}+e^{\frac{\pi\omega}{\alpha}}}{e^{\frac{\pi\omega}{\alpha}}-e^{-\frac{\pi\omega}{\alpha}}} = \frac{i\pi}{\alpha}\coth\left(\frac{\pi\omega}{\alpha}\right) \xRightarrow{(99a)} \tilde{f}_1(\omega) = \frac{i\hbar\beta}{2}\coth\left(\frac{\omega\hbar\beta}{2}\right) \xRightarrow{(98)} K_{th} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{f}_2(\omega)\,e^{-i\omega t}\,d\omega \xRightarrow{(59a)}$$

$$K_{th} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{f}_2(\omega)\,e^{-i\omega t}\,d\omega \,\Big|\, \tilde{f}_2(\omega) = \mathcal{F}[f_2(t)] \overset{(98)}{=} \mathcal{F}[\mathrm{Im}(f(t))] \overset{(59a)}{=} \mathcal{F}[\Gamma(t)] = \tilde{\Gamma}(\omega) \overset{(60h)}{=} \frac{1}{2c}(\tilde{G}_+ - \tilde{G}_-)$$

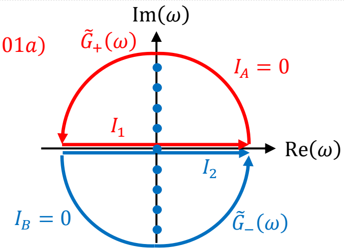
$$\boxed{K_{th} = \frac{i}{4\pi c}\int_{-\infty}^{\infty}\coth\left(\frac{\omega\hbar\beta}{2}\right)(\tilde{G}_+(\omega) - \tilde{G}_-(\omega))e^{-i\omega t}\,d\omega} \ldots (100)$$

… same result for $\omega < 0$, therefore valid for all $\omega \in \mathbb{R}$

Calculate K_{th} with residue theorem.

Matsubara frequencies

$$K_{th} = \frac{i}{4\pi c}\left(\underbrace{\int_{-\infty}^{\infty}\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_+(\omega)\,e^{-i\omega t}\,d\omega}_{I_1} - \underbrace{\int_{-\infty}^{\infty}\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_-(\omega)\,e^{-i\omega t}\,d\omega}_{I_2}\right) \ldots (101a)$$



$$K_{th} = \frac{i}{4\pi c}(I_1 - I_2) \ldots (101b)$$

$\tilde{G}_+(\omega)$ decays in the upper-half plane $\Longrightarrow$ close with I_A in the upper half plane[1)]

$\tilde{G}_-(\omega)$ decays in the lower-half plane $\Longrightarrow$ close with I_B in the lower half plane

$\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_\pm(\omega)\,e^{-i\omega t}$ has poles where $\coth\left(\frac{\omega\hbar\beta}{2}\right) = \frac{\cosh\left(\frac{\omega\hbar\beta}{2}\right)}{\sinh\left(\frac{\omega\hbar\beta}{2}\right)} \to \infty \Longrightarrow$

$$\sinh\left(\frac{\omega\hbar\beta}{2}\right) = -i\sin\left(i\frac{\omega\hbar\beta}{2}\right) = 0 \Longrightarrow i\frac{\omega\hbar\beta}{2} = m\pi \Longrightarrow \boxed{\omega_m = \pm i\frac{2\pi}{\hbar\beta}m \text{ with } m \in \mathbb{N}} \ldots (101c)$$

Matsubara frequencies: $\boxed{\xi_m \overset{\text{def}}{=} \frac{2\pi}{\hbar\beta}m} \ldots (101d) \xRightarrow{(101c)} \boxed{\omega_m = \pm i\xi_m \text{ with } m \in \mathbb{N}} \ldots (101e)$

$$\mathrm{Res}_{\omega\to 0} = \lim_{\omega\to 0}\omega\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_\pm(\omega)\,e^{-i\omega t} = \lim_{\omega\to 0}\frac{\omega}{\tanh\left(\frac{\omega\hbar\beta}{2}\right)}\tilde{G}_\pm(\omega)\,e^{-i\omega t} = \lim_{\omega\to 0}\frac{\partial_\omega\omega}{\partial_\omega\tanh\left(\frac{\omega\hbar\beta}{2}\right)}\tilde{G}_\pm(\omega)\,e^{-i\omega t} \Longrightarrow$$

$$= \lim_{\omega\to 0}\frac{1}{\frac{\hbar\beta}{2}\frac{1}{\cosh^2\left(\frac{\omega\hbar\beta}{2}\right)}}\tilde{G}_\pm(\omega)\,e^{-i\omega t} \Longrightarrow \boxed{\mathrm{Res}_{\omega\to 0} = \frac{2}{\hbar\beta}\tilde{G}_\pm(0)} \ldots (101f)$$

$$\mathrm{Res}_{\omega\to\pm i\xi_m} = \lim_{\omega\to\pm i\xi_m}(\omega \mp i\xi_m)\coth\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_\pm(\omega)\,e^{-i\omega t} = \lim_{\omega\to\pm i\xi_m}\frac{\omega\mp i\xi_m}{\tanh\left(\frac{\omega\hbar\beta}{2}\right)}\tilde{G}_\pm(\omega)\,e^{-i\omega t} \Longrightarrow$$

$$= \lim_{\omega\to\pm i\xi_m}\frac{\partial_\omega(\omega\mp i\xi_m)}{\partial_\omega\tanh\left(\frac{\omega\hbar\beta}{2}\right)}\tilde{G}_\pm(\omega)\,e^{-i\omega t} = \lim_{\omega\to\pm i\xi_m}\frac{1}{\frac{\hbar\beta}{2}\frac{1}{\cosh^2\left(\frac{\omega\hbar\beta}{2}\right)}}\tilde{G}_\pm(\omega)\,e^{-i\omega t} = \lim_{\omega\to\pm i\xi_m}\frac{2}{\hbar\beta}\cosh^2\left(\frac{\omega\hbar\beta}{2}\right)\tilde{G}_\pm(\omega)\,e^{-i\omega t}$$

$$= \frac{2}{\hbar\beta}\cosh^2\left(\pm i\xi_m\frac{\hbar\beta}{2}\right)\tilde{G}_\pm(\pm i\xi_m)\,e^{-i(\pm i\xi_m)t} \overset{(101d)}{=} \frac{2}{\hbar\beta}\cosh^2(\pm i\,\pi\,m)\,\tilde{G}_\pm(\pm i\xi_m)\,e^{\pm\xi_m t} = \frac{2}{\hbar\beta}\tilde{G}_\pm(\pm i\xi_m)\,e^{\pm\xi_m t} \Longrightarrow$$

$$\boxed{\mathrm{Res}_{\omega\to\pm i\xi_m} = \frac{2}{\hbar\beta}\tilde{G}_\pm(\pm i\xi_m)\,e^{\pm\xi_m t}} \ldots (101g)$$

Use Residue Theorem (only half the value for the residue at $\omega = 0$, because we integrate directly through it):

$$\left.\begin{aligned} I_1 + \underbrace{I_A}_{0} &= \pi i\,\mathrm{Res}_{\omega\to 0} + 2\pi i\sum_{m=1}^{\infty}\mathrm{Res}_{\omega\to i\xi_m} \xRightarrow{(101fg)} I_1 = \pi i\frac{2}{\hbar\beta}\tilde{G}_+(0) + 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}_+(i\xi_m)\,e^{\xi_m t} \\ -I_2 + \underbrace{I_B}_{0} &= \pi i\,\mathrm{Res}_{\omega\to 0} + 2\pi i\sum_{m=1}^{\infty}\mathrm{Res}_{\omega\to -i\xi_m} \xRightarrow{(101fg)} I_2 = -\pi i\frac{2}{\hbar\beta}\tilde{G}_-(0) - 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}_-(-i\xi_m)\,e^{-\xi_m t} \end{aligned}\right\} \xRightarrow{(101b)}$$

$$K_{th} = \frac{i}{4\pi c}\left(\pi i\frac{2}{\hbar\beta}\tilde{G}_+(0) + 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}_+(i\xi_m)\,e^{\xi_m t} + \pi i\frac{2}{\hbar\beta}\tilde{G}_-(0) + 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}_-(-i\xi_m)\,e^{-\xi_m t}\right) \xRightarrow{(60g)}$$

$$K_{th} = \frac{i}{4\pi c}\left(\pi i\frac{2}{\hbar\beta}\tilde{G}(0) + 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\,e^{\xi_m t} + \pi i\frac{2}{\hbar\beta}\tilde{G}(0) + 2\pi i\frac{2}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\,e^{-\xi_m t}\right) \Longrightarrow$$

$$K_{th} = \frac{i}{4\pi c}\left(\pi i\frac{4}{\hbar\beta}\tilde{G}(0) + \pi i\frac{4}{\hbar\beta}\sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\left(e^{\xi_m t} + e^{-\xi_m t}\right)\right) = -\frac{1}{\hbar\beta c}\left(\tilde{G}(0) + \sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\,2\frac{1}{2}\left(e^{\xi_m t} + e^{-\xi_m t}\right)\right)$$

$$K_{th} = -\frac{1}{\hbar\beta c}\left(\tilde{G}(0) + 2\sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\cosh(\xi_m t)\right) \Longrightarrow \boxed{K_{th}(t) = -\frac{2}{\hbar\beta c}\left(\frac{\tilde{G}(0)}{2} + \sum_{m=1}^{\infty}\tilde{G}(i\xi_m)\cosh(\xi_m t)\right)} \ldots (102)$$

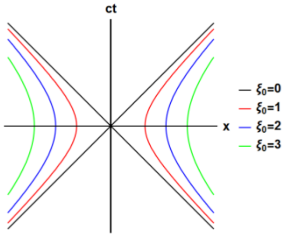
[1)] $\tilde{G}_+(\omega)$ decays in the upper half plane depending on r, but $e^{-i\omega t}$ increases depending on t. Therefore, the integral only converges for $ct < r$.

Low temperature limit	$(94b) \Longrightarrow \beta = \frac{1}{k_B T} \xRightarrow{(101d)} \xi_m = \frac{2\pi}{\hbar} k_B T m \Longrightarrow$ The lower the temperature, the closer are the Matsubara Frequencies $\xRightarrow{(102)}$ Note that $\lim_{\Delta m \to 0} \sum_{m=0}^{\infty} f(\Delta m\, m)\, \Delta m = \int_0^\infty f(\xi)\, d\xi \Longrightarrow \boxed{\lim_{\Delta m\to 0}\sum_{m=0}^\infty f(\Delta m\, m) = \frac{1}{\Delta m}\int_0^\infty f(\xi)\,d\xi}$ ... (103) $f(\zeta) \overset{\text{def}}{=} \tilde G(i\zeta)\cosh(\zeta t) \xRightarrow{(103)} \lim_{\Delta m\to 0}\sum_{m=0}^\infty \tilde G(i\,\Delta m\, m)\cosh(\Delta m\, m\, t) = \frac{1}{\Delta m}\int_0^\infty \tilde G(i\xi)\cosh(\xi t)\,d\xi \,\Big\vert\, \Delta m \overset{\text{def}}{=} \frac{2\pi}{\hbar\beta}$ $\lim_{\frac{2\pi}{\hbar\beta}\to 0}\sum_{m=0}^\infty \tilde G\left(i\frac{2\pi}{\hbar\beta}m\right)\cosh\left(\frac{2\pi}{\hbar\beta}m\,t\right) = \frac{\hbar\beta}{2\pi}\int_0^\infty \tilde G(i\xi)\cosh(\xi t)\,d\xi \,\Big\vert\, \frac{2\pi}{\hbar\beta}\to 0 \Leftrightarrow T\to 0,\ \frac{2\pi}{\hbar\beta}m = \xi_m$ $\lim_{T\to0}\sum_{m=0}^\infty \tilde G(i\xi_m)\cosh(\xi_m\,t) = \frac{\hbar\beta}{2\pi}\int_0^\infty\tilde G(i\xi)\cosh(\xi t)\,d\xi \,\Big\vert$ cf. (102) $\Longrightarrow \lim_{T\to0}K_{th} = -\frac{2}{\hbar\beta c}\frac{\hbar\beta}{2\pi}\int_0^\infty \tilde G(i\xi)\cosh(\xi t)\,d\xi \Longrightarrow$ $\boxed{\lim_{T\to0}K_{th}(t) = -\frac{1}{\pi c}\int_0^\infty\tilde G(i\xi)\cosh(\xi t)\,d\xi}$ ... (104)
Example: 3D space with thermal radiation	Green's Function, spherical wave: $\boxed{\tilde G = -\frac{1}{4\pi r}e^{ikr} = -\frac{1}{4\pi r}e^{i\frac{\omega}{c}r}}$ ... (105) $\xRightarrow{(102)}$ $K_{th} = \frac{2}{\hbar\beta c}\left(-\frac{1}{2}\frac{1}{4\pi r} - \frac{1}{4\pi r}\sum_{m=1}^\infty e^{-\xi_m\frac{r}{c}}\cosh(\xi_m t)\right) = \frac{1}{2\hbar\beta c\pi r}\left(-\frac{1}{2} - \sum_{m=1}^\infty e^{-\xi_m\frac{r}{c}}\cosh(\xi_m t)\right)$ $= \frac{1}{2\hbar\beta c\pi r}\left(-\frac12 - \sum_{m=0}^\infty e^{-\xi_m\frac{r}{c}}\cosh(\xi_m t) + 1\right) = \frac{1}{2\hbar\beta c\pi r}\left(\frac12 - \sum_{m=0}^\infty e^{-\xi_m\frac rc}\cosh(\xi_m t)\right)$ $= -\frac{1}{4\hbar\beta c\pi r}\left(2\sum_{m=0}^\infty e^{-\xi_m\frac rc}\cosh(\xi_m t) - 1\right) = -\frac{1}{4\hbar\beta c\pi r}\left(2\sum_{m=0}^\infty e^{-\xi_m\frac rc}\frac12\left(e^{\xi_m t}+e^{-\xi_m t}\right) - 1\right)$ $= -\frac{1}{4\hbar\beta c\pi r}\left(\sum_{m=0}^\infty\left(e^{\xi_m t}e^{-\xi_m\frac rc} + e^{-\xi_m t}e^{-\xi_m\frac rc}\right) - 1\right) = -\frac{2\pi}{\hbar\beta}\frac{1}{8\pi^2 cr}\left(\sum_{m=0}^\infty\left(e^{\xi_m\left(t-\frac rc\right)} + e^{-\xi_m\left(t+\frac rc\right)}\right)-1\right) \xRightarrow{(101d)}$ $= -\xi_1\frac{1}{8\pi^2 cr}\left(\sum_{m=0}^\infty\left(e^{\xi_1\left(t-\frac rc\right)m} + e^{-\xi_1\left(t+\frac rc\right)m}\right)-1\right) = -\frac{\xi_1}{8\pi^2 cr}\left(\sum_{m=0}^\infty\left(e^{\xi_1\left(t-\frac rc\right)}\right)^m + \sum_{m=0}^\infty\left(e^{-\xi_1\left(t+\frac rc\right)}\right)^m - 1\right)$ $= -\frac{\xi_1}{8\pi^2cr}\left(\frac{1}{1-e^{\xi_1\left(t-\frac rc\right)}} + \frac{1}{1-e^{-\xi_1\left(t+\frac rc\right)}} - 1\right) = -\frac{\xi_1}{8\pi^2cr}\left(\frac{1}{1-e^{\xi_1\left(t-\frac rc\right)}} + \frac{1-\left(1-e^{-\xi_1\left(t+\frac rc\right)}\right)}{1-e^{-\xi_1\left(t+\frac rc\right)}}\right) = -\frac{\xi_1}{8\pi^2cr}\left(\frac{1}{1-e^{\xi_1\left(t-\frac rc\right)}} + \frac{e^{-\xi_1\left(t+\frac rc\right)}}{1-e^{-\xi_1\left(t+\frac rc\right)}}\right)$ $= -\frac{\xi_1}{8\pi^2cr}\left(\frac{1}{1-e^{\xi_1 t}e^{-\xi_1 r/c}}\cdot\frac{e^{\xi_1 r/c}}{e^{\xi_1 r/c}} + \frac{e^{-\xi_1 t}e^{-\xi_1 r/c}}{1-e^{-\xi_1 t}e^{-\xi_1 r/c}}\cdot\frac{e^{\xi_1 t}}{e^{\xi_1 t}}\right) = -\frac{\xi_1}{8\pi^2cr}\left(\frac{e^{\xi_1 r/c}}{e^{\xi_1 r/c}-e^{\xi_1 t}} + \frac{e^{-\xi_1 r/c}}{e^{\xi_1 t}-e^{-\xi_1 r/c}}\right) \Longrightarrow$ $\boxed{K_{th} = -\frac{\xi_1}{8\pi^2cr}\left(\frac{e^{\xi_1 r/c}}{e^{\xi_1 r/c}-e^{\xi_1 t}} - \frac{e^{-\xi_1 r/c}}{e^{-\xi_1 r/c}-e^{\xi_1 t}}\right)}$ ... (106a) $\boxed{u \overset{\text{def}}{=} e^{\xi_1 t},\ u_+ \overset{\text{def}}{=} e^{\xi_1 r/c},\ u_- \overset{\text{def}}{=} e^{-\xi_1 r/c}}$ ... (106b) $\xRightarrow{(106a)}$ $\boxed{K_{th} = -\frac{\xi_1}{8\pi^2cr}\left(\frac{u_+}{u_+-u} - \frac{u_-}{u_--u}\right)}$ ... (106c)
	Note that $\boxed{K_{th} = -\frac{1}{8\pi^2 r}\partial_r\ln\big((u-u_+)(u-u_-)\big)}$ ... (107) Proof: $K_{th} = -\frac{1}{8\pi^2r}\partial_r\ln\big((u-u_+)(u-u_-)\big) = -\frac{1}{8\pi^2r}\frac{\partial_r\{[u(t)-u_+(r)][u(t)-u_-(r)]\}}{(u-u_+)(u-u_-)} = -\frac{1}{8\pi^2r}\frac{\partial_r\left(u^2(t)+u_+(r)\,u_-(r)-u(t)\,u_-(r)-u(t)\,u_+(r)\right)}{(u-u_+)(u-u_-)}$ $= -\frac{1}{8\pi^2r}\frac{(\partial_r u_+(r))\,u_-(r)+u_+(r)\,\partial_r u_-(r)-u(t)\,\partial_r u_-(r)-u(t)\,\partial_r u_+(r)}{(u_--u_+)(u-u_-)} \overset{(106b)}{=} -\frac{1}{8\pi^2r}\frac{(\partial_r e^{\xi_1 r/c})e^{-\xi_1 r/c}+e^{\xi_1 r/c}\partial_r e^{-\xi_1 r/c}-e^{\xi_1 t}\partial_r e^{-\xi_1 r/c}-e^{\xi_1 t}\partial_r e^{\xi_1 r/c}}{(e^{\xi_1 t}-e^{\xi_1 r/c})(e^{\xi_1 t}-e^{-\xi_1 r/c})}$ $= -\frac{1}{8\pi^2r}\frac{\cancel{\frac{\xi_1}{c}e^{\xi_1 r/c}e^{-\xi_1 r/c}-\frac{\xi_1}{c}e^{\xi_1 r/c}e^{-\xi_1 r/c}}+\frac{\xi_1}{c}e^{\xi_1 t}e^{-\xi_1 r/c}-\frac{\xi_1}{c}e^{\xi_1 t}e^{\xi_1 r/c}}{(e^{\xi_1 t}-e^{\xi_1 r/c})(e^{\xi_1 t}-e^{-\xi_1 r/c})} = -\frac{\xi_1}{8\pi^2cr}\frac{e^{\xi_1 t}e^{-\xi_1 r/c}-e^{\xi_1 t}e^{\xi_1 r/c}}{(e^{\xi_1 t}-e^{\xi_1 r/c})(e^{\xi_1 t}-e^{-\xi_1 r/c})}$ $= -\frac{\xi_1}{8\pi^2cr}\frac{-e^{\xi_1 t}e^{\xi_1 r/c}+e^{\xi_1 t}e^{-\xi_1 r/c}}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})} = -\frac{\xi_1}{8\pi^2cr}\frac{1-e^{\xi_1 t}e^{\xi_1 r/c}-1+e^{\xi_1 t}e^{-\xi_1 r/c}}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})} = -\frac{\xi_1}{8\pi^2cr}\frac{e^{\xi_1 r/c}e^{-\xi_1 r/c}-e^{\xi_1 t}e^{\xi_1 r/c}-e^{-\xi_1 r/c}e^{\xi_1 r/c}+e^{\xi_1 t}e^{-\xi_1 r/c}}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})}$ $= -\frac{\xi_1}{8\pi^2cr}\frac{e^{\xi_1 r/c}(e^{-\xi_1 r/c}-e^{\xi_1 t})-e^{-\xi_1 r/c}(e^{\xi_1 r/c}-e^{\xi_1 t})}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})} = -\frac{\xi_1}{8\pi^2cr}\left(\frac{e^{\xi_1 r/c}(e^{-\xi_1 r/c}-e^{\xi_1 t})}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})} - \frac{e^{-\xi_1 r/c}(e^{\xi_1 r/c}-e^{\xi_1 t})}{(e^{\xi_1 r/c}-e^{\xi_1 t})(e^{-\xi_1 r/c}-e^{\xi_1 t})}\right) \Longrightarrow$ $K_{th} = -\frac{\xi_1}{8\pi^2cr}\left(\frac{e^{\xi_1 r/c}}{e^{\xi_1 r/c}-e^{\xi_1 t}} - \frac{e^{-\xi_1 r/c}}{e^{-\xi_1 r/c}-e^{\xi_1 t}}\right)$ ✓cf. $(106a)$ Interpretation: $u \overset{\text{def}}{=} e^{\xi_1 t}$ from $(106b)$ corresponds to the exponential map $w = e^{\frac{2\pi}{\hbar\beta}z}$ (with z ...complex time), introduced in $(96a)$. The exponential map is also applied to $\pm r/c$ because the time-dependent Green's function of empty 3D space is $G_\pm = -\frac{1}{4\pi r}\delta(t \mp r/c)$. The poles $u_\pm$ in the correlation function K correspond to the δ-peaks in the Green's function at $t = \pm\frac{r}{ct}$.
Small temperature limit	For small temperatures: $\lvert\xi_1 t\rvert \ll 1$ and $\left\lvert\frac{\xi_1 r}{c}\right\rvert \ll 1. \xRightarrow{(106b)} \boxed{u \approx 1+\xi_1 t,\ u_+ \approx 1+\frac{\xi_1 r}{c},\ u_- \approx 1-\frac{\xi_1 r}{c}}$ ... (108) $\xRightarrow{(107)}$ $\lim_{T\to0}K_{th} = -\frac{1}{8\pi^2r}\partial_r\ln\left(\left(1+\xi_1 t-1-\frac{\xi_1 r}{c}\right)\left(1+\xi_1 t-1+\frac{\xi_1 r}{c}\right)\right) = -\frac{1}{8\pi^2r}\partial_r\ln\left(\left(\xi_1 t-\frac{\xi_1 r}{c}\right)\left(\xi_1 t+\frac{\xi_1 r}{c}\right)\right)$ $= -\frac{1}{8\pi^2r}\partial_r\ln\left(\xi_1^2t^2-\frac{\xi_1^2r^2}{c^2}\right) = -\frac{1}{8\pi^2r}\partial_r\ln\left(\frac{\xi_1^2}{c^2}(t^2c^2-r^2)\right) = \frac{1}{8\pi^2r}\frac{\frac{\xi_1^2}{c^2}2r}{\frac{\xi_1^2}{c^2}(c^2t^2-r^2)} \Longrightarrow \boxed{\lim_{T\to0}K_{th} = \frac{1}{4\pi^2(t^2c^2-r^2)}}$ ... (109) Minkowsky metric: $\boxed{s^2 = c^2t^2-r^2}$ ... (110) ... spacetime distance between emission point r_1, t_1 and reception point r_2, t_2 $\xRightarrow{(109)} \boxed{\lim_{T\to0}K_{th} = \frac{1}{4\pi^2s^2}}$ ... (111) ... vacuum term. In renormalization, this term is discarded. Expanding expression (107) for small ξ_1 yields: $\boxed{K_{th} \approx \frac{1}{4\pi^2s^2} + \frac{\xi_1^2}{48\pi^2c^2} + \mathcal{O}(\xi_1^4)}$ ... (112) The first term in expression (112) is the vacuum term from (111). The second term (and all other following terms) are thermal effects, which depend on the temperature. For $t = 0$ the third term is proportional to ξ_1^4, resulting in an energy density $\epsilon \propto \partial_r^2 K \propto \xi_1^4 \propto T^4$ (Stefan Boltzmann Law)

3 Acceleration and Expansion

3.1 Uniform Acceleration

3.1.1 Rindler Coordinates

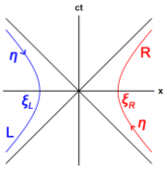
Reminder: Metric and Proper Time	Assumption: 1+1 dim Minkowsky Space. The length element ds^2 is determined by the metric: $\boxed{ds^2 = c^2\,dt^2 - dx^2}$ … (113) ds^2 is invariant under Lorentz-Transformation (LT). In the system, to which the observer is at rest, we can write: $dx^2 = 0$. Therefore, the time this observer experiences (called Proper Time $d\tau$) is determined by $ds^2 = c^2\,d\tau^2 \Rightarrow \boxed{d\tau = \frac{ds}{c}}$ … (114)
Rindler Coordinates in Flat Space	1+1 dimensional Minkowsky Space. Hyperbolic polar coordinates (“Rindler Coordinates”): $\boxed{x = \xi\cosh(\eta)}$ … $(115a)$ $\boxed{ct = \xi\sinh(\eta)}$ … $(115b) \Longrightarrow \underline{t = \frac{1}{c}\xi\sinh(\eta)}$ … $(115c)$  Lines in the spacetime diagram, where the invariant metric length element s equals a specific constant length $\xi = \xi_0$, are hyperbolas (because of $s \overset{!}{=} \xi_0 \Longrightarrow c^2t^2 - x^2 = \xi_0$) $\underline{\frac{\partial x}{\partial\xi} \overset{(115a)}{=} \cosh(\eta)}$ … $(116a)$ $\underline{\frac{\partial x}{\partial\eta} \overset{(115a)}{=} \xi\sinh(\eta)}$ … $(116b)$ $\underline{\frac{\partial t}{\partial\xi} \overset{(115c)}{=} \frac{1}{c}\sinh(\eta)}$ … $(116c)$ $\underline{\frac{\partial t}{\partial\eta} \overset{(115a)}{=} \frac{1}{c}\xi\cosh(\eta)}$ … $(116d)$ $\left.\begin{array}{l} dx = \frac{\partial x}{\partial\xi}d\xi + \frac{\partial x}{\partial\eta}d\eta \overset{(116ab)}{=} \cosh(\eta)\,d\xi + \xi\sinh(\eta)\,d\eta \Longrightarrow \\ \underline{dx^2 = \cosh^2(\eta)\,d\xi^2 + \xi^2\sinh^2(\eta)\,d\eta^2 + 2\sinh(\eta)\cosh(\eta)\,d\xi\,d\eta} \text{ … } (117a) \\ dt = \frac{\partial t}{\partial\xi}d\xi + \frac{\partial t}{\partial\eta}d\eta \overset{(116cd)}{=} \frac{1}{c}\sinh(\eta)\,d\xi + \frac{1}{c}\xi\cosh(\eta)\,d\eta \Longrightarrow c\,dt = \sinh(\eta)\,d\xi + \xi\cosh(\eta)\,d\eta \Longrightarrow \\ \underline{c^2\,dt^2 = \sinh^2(\eta)\,d\xi^2 + \xi^2\cosh^2(\eta)\,d\eta^2 + 2\sinh(\eta)\cosh(\eta)\,d\xi\,d\eta} \text{ … } (117b) \end{array}\right\} \overset{(113)}{\Longrightarrow}$ $ds^2 = (\sinh^2(\eta) - \cosh^2(\eta))\,d\xi^2 + \xi^2(\cosh^2(\eta) - \sinh^2(\eta))\,d\eta^2 + \cancel{2\sinh(\eta)\cosh(\eta)\,d\xi\,d\eta - 2\sinh(\eta)\cosh(\eta)\,d\xi\,d\eta}$ $ds^2 = \xi^2(\cosh^2(\eta) - \sinh^2(\eta))\,d\eta^2 - (\cosh^2(\eta) - \sinh^2(\eta))\,d\xi^2 \,\vert\, \cosh^2(\eta) - \sinh^2(\eta) = 1 \Rightarrow \boxed{ds^2 = \xi^2\,d\eta^2 - d\xi^2}$ … (118)
Rindler Trajectory	For an observer moving along a hyperbola $c^2t^2 - x^2 = \xi_0$ the Rindler coordinate $\xi = \xi_{\text{const}} \Longrightarrow d\xi^2 = 0 \overset{(118)}{\Longrightarrow}$ $ds^2 = \xi_{\text{const}}^2\,d\eta^2 \Longrightarrow \underline{ds = \xi_{\text{const}}\,d\eta}$ … (119) The observer on a Rindler Trajectory experiences the following proper time: $\tau = \int d\tau \overset{(114)}{=} \frac{1}{c}\int ds \overset{(119)}{=} \frac{1}{c}\int \xi_{\text{const}}\,d\eta \Longrightarrow \boxed{\tau = \frac{\xi_{\text{const}}}{c}\eta}$ … (120) So, for an observer with fixed ξ_{const} the Proper Time $\tau \propto \eta$ Velocity w.r.t. to Proper Time: $u = \frac{dx}{d\tau} = \frac{\partial x/\partial\eta}{\partial\tau/\partial\eta} \overset{(116b)}{=} \frac{\xi_0\sinh(\eta)}{\partial\tau/\partial\eta}\Big\vert\, \frac{\partial\tau}{\partial\eta} \overset{(120)}{=} \frac{\xi_0}{c} \Rightarrow u = c\sinh(\eta) \Longrightarrow \xi_{\text{const}}u = c\,\xi_{\text{const}}\sinh(\eta) \overset{(115b)}{\Longrightarrow}$ $\xi_{\text{const}}u = c^2t \Longrightarrow \boxed{u = \frac{c^2}{\xi_{\text{const}}}t}$ … (121) Acceleration for an observer on a Rindler Trajectory: $a = \frac{du}{dt} \overset{(121)}{\Longrightarrow} \boxed{a = \frac{c^2}{\xi_{\text{const}}} = \text{const.}}$ … (122)

3.1.2 Vacuum Correlations

Vacuum Correlations	The vacuum state $\vert 0\rangle$, so that $\hat{a}_k\vert 0\rangle = \vert 0\rangle$ in Minkowsky Space is defined for plane waves with a specific polarization. Therefore the vector potential can be written as $A_k \propto e^{-k_\mu x^\mu}$. Since $k_\mu \overset{\text{def}}{=} \left(\frac{\omega}{c}, k\right)^T$ and $x_\mu \overset{\text{def}}{=} (ct, x)^T$ are covariant vectors, the vector potential A_k, the Green’s function $\tilde{G}$ and the correlation function K are all Lorentz scalar fields. Therefore, transforming these fields into another coordinate system simply requires transforming the spacetime coordinates. Assuming low temeratures: $\overset{(104)}{\Longrightarrow} K_{th} = \frac{1}{\pi c}\int_0^\infty \tilde{G}(i\xi)\cosh(\xi t)\,d\xi \overset{(75ab)}{=} \frac{1}{\pi}\int_0^\infty \tilde{G}(i\kappa)\cosh(\kappa ct)\,d\kappa \overset{(74)}{\Longrightarrow}$ $K_{th} = -\frac{1}{2\pi}\int_0^\infty \frac{1}{\kappa}e^{-\kappa\vert x\vert}\cosh(\kappa ct)\,d\kappa = -\frac{1}{4\pi}\int_0^\infty \frac{1}{\kappa}e^{-\kappa\vert x\vert}(e^{\kappa ct} + e^{-\kappa ct})\,d\kappa = -\frac{1}{4\pi}\int_0^\infty \frac{1}{\kappa}\left(e^{-\kappa(\vert x\vert - ct)} + e^{-\kappa(\vert x\vert + ct)}\right)d\kappa$ $\partial_{\vert x\vert}K_{th} = -\frac{1}{4\pi}\int_0^\infty \frac{1}{\kappa}\partial_{\vert x\vert}\left(e^{-\kappa(\vert x\vert - ct)} + e^{-\kappa(\vert x\vert + ct)}\right)d\kappa = \frac{1}{4\pi}\int_0^\infty \left(e^{-\kappa(\vert x\vert - ct)} + e^{-\kappa(\vert x\vert + ct)}\right)d\kappa$ $\partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\left(\int_0^\infty e^{-\kappa(\vert x\vert - ct)}\,d\kappa + \int_0^\infty e^{-\kappa(\vert x\vert + ct)}\,d\kappa\right) \left\vert \begin{array}{l} u \overset{\text{def}}{=} -\kappa(\vert x\vert - ct) \Longrightarrow \frac{du}{d\kappa} = -(\vert x\vert - ct) \Longrightarrow d\kappa = -\frac{du}{\vert x\vert - ct} \\ v \overset{\text{def}}{=} -\kappa(\vert x\vert + ct) \Longrightarrow \frac{dv}{d\kappa} = -(\vert x\vert + ct) \Longrightarrow d\kappa = -\frac{dv}{\vert x\vert + ct} \end{array}\right.$ $\partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\left(-\frac{1}{\vert x\vert - ct}\int_{\kappa=0}^{\kappa\to\infty} e^u\,du - \frac{1}{\vert x\vert + ct}\int_{\kappa=0}^{\kappa\to\infty} e^v\,dv\right) = \frac{1}{4\pi}\left(-\frac{1}{\vert x\vert - ct}e^u\Big\vert_{\kappa=0}^{\kappa\to\infty} - \frac{1}{\vert x\vert + ct}e^v\Big\vert_{\kappa=0}^{\kappa\to\infty}\right)$ $\partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\left(-\frac{1}{\vert x\vert - ct}e^{-\kappa(\vert x\vert - ct)}\Big\vert_0^\infty - \frac{1}{\vert x\vert + ct}e^{-\kappa(\vert x\vert + ct)}\Big\vert_0^\infty\right) = \frac{1}{4\pi}\left(-\frac{1}{\vert x\vert - ct}(0 - 1) - \frac{1}{\vert x\vert + ct}(0 - 1)\right) = \frac{1}{4\pi}\left(\frac{1}{\vert x\vert - ct} + \frac{1}{\vert x\vert + ct}\right)$ $\partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\frac{\vert x\vert + ct + \vert x\vert - ct}{(\vert x\vert - ct)(\vert x\vert + ct)} \Longrightarrow \underline{\partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\frac{2\vert x\vert}{\vert x\vert^2 - c^2t^2}}$ … (123) Note that $\partial_{\vert x\vert}\ln(x^2 - c^2t^2) = \partial_{\vert x\vert}\ln(\vert x\vert^2 - c^2t^2) \Longrightarrow$ $\partial_{\vert x\vert}\ln(x^2 - c^2t^2) = \frac{2\vert x\vert}{\vert x\vert^2 - c^2t^2} \overset{(123)}{\Longrightarrow} \partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\,\partial_{\vert x\vert}\ln(x^2 - c^2t^2) \overset{(113)}{\Longrightarrow} \partial_{\vert x\vert}K_{th} = \frac{1}{4\pi}\,\partial_{\vert x\vert}\ln(s^2) = \frac{1}{2\pi}\,\partial_{\vert x\vert}\ln(\vert s\vert) \Longrightarrow$ $K_{th} = \frac{1}{2\pi}\ln(\vert s\vert) + \text{const.}$ … To calculate ϵ, we need to differentiate K_{th} twice, hence the constant does not mater $\Longrightarrow$ $\boxed{K_{\text{eff}} = \frac{1}{2\pi}\ln(\vert s\vert)}$ … (124) with $s^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2$ defining spacetime-distance between emitter and observer. In 1+2 spacetime dimensions: $K_{th} \propto \frac{1}{s}$, In 1+3 spacetime dimensions $K_{th} \propto \frac{1}{s^2}$

Conformal Rindler Coordinates	$\boxed{\xi \overset{\text{def}}{=} e^{\zeta} \Longrightarrow \xi^2 = e^{2\zeta}}$ … (125a) $\Longrightarrow \frac{d\xi}{d\zeta} = e^{\zeta} \Longrightarrow \boxed{d\xi = e^{\zeta}d\zeta \Longrightarrow d\xi^2 = e^{2\zeta}d\zeta^2}$ … (125b) (115ab) $\xrightarrow{(118)}$ $ds^2 = e^{2\zeta}\, d\eta^2 - e^{2\zeta}d\zeta^2 \Longrightarrow \boxed{ds^2 = e^{2\zeta}\left(d\eta^2 - d\zeta^2\right)}$ … (126) global scaling factor $e^{2\zeta}$ … “conformal factor” $\boxed{s^2 \overset{\text{def}}{=} s_-s_+}$ … (127a) $\boxed{s_- = c(t_2-t_1)-(x_2-x_1)}$ … (127b) $\boxed{s_+ = c(t_2-t_1)+(x_2-x_1)}$ … (127c) $x_i \overset{(115a)}{=} \xi_i \cosh(\eta_i) = \frac{1}{2}\xi_i(e^{\eta_i}+e^{-\eta_i}) \xrightarrow{(125a)} \boxed{x_i = \frac{1}{2}e^{\zeta_i}(e^{\eta_i}+e^{-\eta_i})}$ … (128a) $ct_i \overset{(115b)}{=} \xi_i \sinh(\eta_i) = \frac{1}{2}\xi_i(e^{\eta_i}-e^{-\eta_i}) \xrightarrow{(125a)} \boxed{ct_i = \frac{1}{2}e^{\zeta_i}(e^{\eta_i}-e^{-\eta_i})}$ … (128b) $\Big\} \xrightarrow{(127bc)}$ $s_- = \frac{1}{2}e^{\zeta_2}(e^{\eta_2}-e^{-\eta_2}) - \frac{1}{2}e^{\zeta_1}(e^{\eta_1}-e^{-\eta_1}) - \frac{1}{2}e^{\zeta_2}(e^{\eta_2}+e^{-\eta_2}) + \frac{1}{2}e^{\zeta_1}(e^{\eta_1}+e^{-\eta_1}) \Longrightarrow$ $s_- = \cancel{\frac{1}{2}e^{\zeta_2}e^{\eta_2}} - \frac{1}{2}e^{\zeta_2}e^{-\eta_2} - \cancel{\frac{1}{2}e^{\zeta_1}e^{\eta_1}} + \frac{1}{2}e^{\zeta_1}e^{-\eta_1} - \cancel{\frac{1}{2}e^{\zeta_2}e^{\eta_2}} - \frac{1}{2}e^{\zeta_2}e^{-\eta_2} + \cancel{\frac{1}{2}e^{\zeta_1}e^{\eta_1}} + \frac{1}{2}e^{\zeta_1}e^{-\eta_1} \Longrightarrow$ $s_- = -e^{\zeta_2}e^{-\eta_2} + e^{\zeta_1}e^{-\eta_1} \Longrightarrow \boxed{s_- = -e^{\zeta_2-\eta_2} + e^{\zeta_1-\eta_1}}$ … (129a) $s_+ = \frac{1}{2}e^{\zeta_2}(e^{\eta_2}-e^{-\eta_2}) - \frac{1}{2}e^{\zeta_1}(e^{\eta_1}-e^{-\eta_1}) + \frac{1}{2}e^{\zeta_2}(e^{\eta_2}+e^{-\eta_2}) - \frac{1}{2}e^{\zeta_1}(e^{\eta_1}+e^{-\eta_1}) \Longrightarrow$ $s_+ = \frac{1}{2}e^{\zeta_2}e^{\eta_2} - \cancel{\frac{1}{2}e^{\zeta_2}e^{-\eta_2}} - \frac{1}{2}e^{\zeta_1}e^{\eta_1} + \cancel{\frac{1}{2}e^{\zeta_1}e^{-\eta_1}} + \frac{1}{2}e^{\zeta_2}e^{\eta_2} + \cancel{\frac{1}{2}e^{\zeta_2}e^{-\eta_2}} - \frac{1}{2}e^{\zeta_1}e^{\eta_1} - \cancel{\frac{1}{2}e^{\zeta_1}e^{-\eta_1}} \Longrightarrow$ $s_+ = e^{\zeta_2}e^{\eta_2} - e^{\zeta_1}e^{\eta_1} \Longrightarrow \boxed{s_+ = e^{\zeta_2+\eta_2} - e^{\zeta_1+\eta_1}}$ … (129b) (127a) $\Longrightarrow s^2 \overset{\text{def}}{=} s_-s_+ \overset{(129ab)}{=} (-e^{\zeta_2-\eta_2} + e^{\zeta_1-\eta_1})(e^{\zeta_2+\eta_2} - e^{\zeta_1+\eta_1}) = -e^{2\zeta_2} - e^{2\zeta_1} + e^{\zeta_1}e^{\zeta_2}e^{\eta_1}e^{-\eta_2} + e^{\zeta_1}e^{\zeta_2}e^{-\eta_1}e^{\eta_2} \Longrightarrow$ $s^2 = e^{\zeta_1}e^{\zeta_2}e^{\eta_1}e^{-\eta_2}(-e^{-\zeta_1}e^{\zeta_2}e^{-\eta_1}e^{\eta_2} - e^{\zeta_1}e^{-\zeta_2}e^{-\eta_1}e^{\eta_2} + 1 + e^{-2\eta_1}e^{2\eta_2}) \Longrightarrow$ $s^2 = e^{\zeta_1+\zeta_2+\eta_1-\eta_2}\left(-e^{(\zeta_2-\zeta_1)+(\eta_2-\eta_1)} - e^{-(\zeta_2-\zeta_1)+(\eta_2-\eta_1)} + 1 + e^{2(\eta_2-\eta_1)}\right)\Big\vert \zeta \overset{\text{def}}{=} \zeta_2-\zeta_1,\ \eta \overset{\text{def}}{=} \eta_2-\eta_1 \Longrightarrow$ $s^2 = e^{\zeta_1+\zeta_2+\eta_1-\eta_2}(-e^{\zeta+\eta} - e^{-\zeta+\eta} + 1 + e^{2\eta}) \Longrightarrow \boxed{s^2 = e^{\zeta_1+\zeta_2+\eta_1-\eta_2}(e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})}$ … (130a) $\Longrightarrow$ $\boxed{\ln(s^2) = \zeta_1+\zeta_2+\eta_1-\eta_2 \ln\left((e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})\right)}$ … (130b) $\Longrightarrow$ $2\ln(\lvert s\rvert) = \zeta_1+\zeta_2+\eta_1-\eta_2 \ln\left((e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})\right) \Longrightarrow \ln(\lvert s\rvert) = \frac{\zeta_1+\zeta_2+\eta_1-\eta_2 \ln\left((e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})\right)}{2} \xrightarrow{(124)}$ $K_{\text{th}} = \frac{1}{4\pi}\left[\zeta_1+\zeta_2+\eta_1-\eta_2 \ln\left((e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})\right)\right]$ To calculate ϵ and other observables, we need to differentiate K_{th} twice, hence constants and linear terms don’t mater $\Longrightarrow$ $\boxed{K_{\text{eff}} = \frac{1}{4\pi}\ln\left((e^{\eta}-e^{\zeta})(e^{\eta}-e^{-\zeta})\right)}$ … (131)
Unruh Effect	Start with 1D empty space Green’s function (74) $\Longrightarrow \tilde{G} = -\frac{1}{2\kappa}e^{-\kappa\lvert x\rvert} \overset{(75a)}{=} -\frac{c}{2\xi}e^{-\frac{\xi}{c}\lvert x\rvert} \xrightarrow{(102)}$ $K = -\frac{2}{\hbar\beta c}\left(-\frac{1}{2}\frac{c}{2\xi_0} - \sum_{m=1}^{\infty}\frac{c}{2\xi_m}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t)\right) \Longrightarrow \underline{K = -\frac{1}{\hbar\beta c}\left(-\frac{1}{2}\frac{c}{\xi_0}e^{-\frac{\xi_0}{c}\lvert x\rvert} - \sum_{m=1}^{\infty}\frac{c}{\xi_m}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t)\right)}$ … (132) Note that $\int\left(\frac{1}{2}e^{-\frac{\xi_0}{c}\lvert x\rvert} + \sum_{m=1}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t)\right)d\lvert x\rvert = -\frac{1}{2}\frac{c}{\xi_0} - \sum_{m=1}^{\infty}\frac{c}{\xi_m}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t) + \text{const.} \xrightarrow{\text{cf.}(132)}$ $K = -\frac{1}{\hbar\beta c}\int\left(\frac{1}{2}e^{-\frac{\xi_0}{c}\lvert x\rvert} + \sum_{m=1}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t)\right)d\lvert x\rvert - \text{const.}$ To calculate ϵ and other observables, we need to differentiate K_{th} twice, hence constants don’t mater. Also, this constant helps to eliminate the divergency from the $-\frac{1}{2}\frac{c}{\xi_0} \to \infty$ term (remember that $\xi_0 = 0$). $\Longrightarrow$ $K_{\text{eff}} = -\frac{1}{\hbar\beta c}\int\left(\frac{1}{2}e^{-\frac{\xi_0}{c}\lvert x\rvert} + \sum_{m=1}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t)\right)d\lvert x\rvert = -\frac{1}{\hbar\beta c}\int\left(\sum_{m=0}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t) - \frac{1}{2}\right)d\lvert x\rvert$ $= -\frac{1}{2\hbar\beta c}\int\left(2\sum_{m=0}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\cosh(\xi_m t) - 1\right)d\lvert x\rvert = -\frac{1}{2\hbar\beta c}\int\left(2\sum_{m=0}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}\frac{1}{2}\left(e^{\xi_m t} + e^{-\xi_m t}\right) - 1\right)d\lvert x\rvert$ $= -\frac{1}{2\hbar\beta c}\int\left(\sum_{m=0}^{\infty}e^{-\frac{\xi_m}{c}\lvert x\rvert}e^{\xi_m t} + e^{-\frac{\xi_m}{c}\lvert x\rvert}e^{-\xi_m t} - 1\right)d\lvert x\rvert = -\frac{2\pi}{\hbar\beta}\frac{1}{4\pi c}\int\left(\sum_{m=0}^{\infty}e^{\xi_m\left(t-\frac{\lvert x\rvert}{c}\right)} + e^{-\xi_m\left(t+\frac{\lvert x\rvert}{c}\right)} - 1\right)d\lvert x\rvert \xrightarrow{(101d)}$ $= -\xi_1\frac{1}{4\pi c}\int\left(\sum_{m=0}^{\infty}e^{\xi_1\left(t-\frac{\lvert x\rvert}{c}\right)m} + e^{-\xi_1\left(t+\frac{\lvert x\rvert}{c}\right)m} - 1\right)d\lvert x\rvert = -\frac{\xi_1}{4\pi c}\int\left(\sum_{m=0}^{\infty}\left(e^{\xi_1\left(t-\frac{\lvert x\rvert}{c}\right)}\right)^m + \sum_{m=0}^{\infty}\left(e^{-\xi_1\left(t+\frac{\lvert x\rvert}{c}\right)}\right)^m - 1\right)d\lvert x\rvert$ $= -\frac{\xi_1}{4\pi c}\int\left(\sum_{m=0}^{\infty}\left(e^{\xi_1\left(t-\frac{\lvert x\rvert}{c}\right)}\right)^m + \sum_{m=0}^{\infty}\left(e^{-\xi_1\left(t+\frac{\lvert x\rvert}{c}\right)}\right)^m - 1\right)d\lvert x\rvert = -\frac{\xi_1}{4\pi c}\int\left(\frac{1}{1-e^{\xi_1\left(t-\frac{\lvert x\rvert}{c}\right)}} + \frac{1}{1-e^{-\xi_1\left(-t+\frac{\lvert x\rvert}{c}\right)}} - 1\right)d\lvert x\rvert \xrightarrow{\text{Mathematica}}$ $= -\frac{1}{4\pi}\ln\left((1-e^{\xi_1(t-\lvert x\rvert/c)})(1-e^{\xi_1(t+\lvert x\rvert/c)})\right) = -\frac{1}{4\pi}\ln\left((1-e^{\xi_1 t}e^{-\xi_1\lvert x\rvert/c})(1-e^{\xi_1 t}e^{\xi_1\lvert x\rvert/c})\right)$ $= -\frac{1}{4\pi}\ln(1+e^{2\xi_1 t} - e^{\xi_1 t}e^{\xi_1\lvert x\rvert/c} - e^{\xi_1 t}e^{-\xi_1\lvert x\rvert/c}) \Longrightarrow K_{\text{eff}} = -\frac{1}{4\pi}\ln\left((e^{\xi_1 t} - e^{\xi_1\lvert x\rvert/c})(e^{\xi_1 t} - e^{-\xi_1\lvert x\rvert/c})\right)$ … (133) By comparing (133) with (131) (sign to be clarified) we can deduce: $\underline{\eta = \xi_1 t}$ … (134a) (120) $\Longrightarrow \tau = \frac{\xi_{\text{const}}}{c}\eta \Longrightarrow \eta = \frac{c}{\xi_{\text{const}}}\tau \xrightarrow{(134a)} \frac{c}{\xi_{\text{const}}}\tau = \xi_1 t \Longrightarrow \frac{c}{\xi'_{\text{const}}}t = \xi_1 t \Longrightarrow \underline{\frac{c}{\xi'_{\text{const}}} = \xi_1}$ … (134b) (122) $\Longrightarrow a = \frac{c^2}{\xi_{\text{const}}} \Longrightarrow \frac{1}{\xi_{\text{const}}} = \frac{a}{c^2} \xrightarrow{(134b)} \frac{a}{c} = \xi_1 \xrightarrow{(101d)} \frac{a}{c} = \frac{2\pi}{\hbar\beta} \xrightarrow{(94b)} \frac{a}{c} = \frac{2\pi}{\hbar}k_B T \Longrightarrow \boxed{k_B T = \frac{\hbar a}{2\pi c}}$ … (135) … Unruh Effect An uniformly accelerated observer experiences thermal radiation of this temperature. However, to reach e.g. room temperature (300K), the required acceleration would be in the magnitude of $a \approx 10^{23}\frac{m}{s^2}$. Therefore, no direct experimental evidence could yet be gathered. However, an analogue of this effect can be measured using a classical system with water waves [Leonhard et al, Phys. Rev. A **98**, 022118].

3.1.2 A Tale of Two Observers

Two Rindler Observers	We consider two Rindler observers L and R along Rindler trajectories, so that $\boxed{\xi_L = -\xi_R < 0}$ ... $(136a)$ Both observers are equipped with spectrometers and measure the Fourier transform of the electromagnetic field with respect to their respective proper time τ. We assume that all modes $\vec{A}_k$ have positive frequencies $k \geq 0$, so that $\boxed{(\vec{A}_k, \vec{A}_k) \geq 0}$ ... $(136b)$ cf. $(25ab)$. Note that for the left observer the proper time τ runs against η. 
Modes	We have to consider both right- and left-moving waves, indicated by indices $+$ and $-$. Plane waves can be expressed in Minkowsky coordinates, respectively Rindler coordinates, as follows: $\boxed{A_{k\pm} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp(ik(\pm x - ct))}$ ... $(137a)$ $\xRightarrow{(115ab)} A_{k\pm} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp(ik(\pm\xi\cosh(\eta) - \xi\sinh(\eta))) \Longrightarrow$ $A_{k\pm} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp(ik\xi(\pm\cosh(\eta) - \sinh(\eta))) = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp\left(ik\xi\left(\pm\frac{1}{2}(e^{\eta} + e^{-\eta}) - \frac{1}{2}(e^{\eta} - e^{-\eta})\right)\right)$ $= \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp\left(ik\xi\frac{1}{2}(\pm e^{\eta} \pm e^{-\eta} - e^{\eta} + e^{-\eta})\right) \Longrightarrow \boxed{A_{k\pm} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\exp(\pm ik\xi e^{\mp\eta})}$ ... $(137b)$
Fourier Transformed Modes, in general	Fourier transform of the Mode $A_{k\pm}$ for a Rindler observer along a Rindler trajectory with constant ξ w.r.t. proper time τ: $\widetilde{A_{k\pm}}\big\rvert_{\xi} \overset{(60a)}{=} \int_{-\infty}^{\infty} A_{k\pm}e^{i\omega\tau}\,d\tau \overset{(137b)}{=} \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\int_{-\infty}^{\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{i\omega\tau}\,d\tau \Big\vert\ (120) \Longrightarrow \tau = \frac{\xi}{c}\eta \Longrightarrow d\tau = \frac{\xi}{c}d\eta,\ \ \eta = \frac{c}{\xi}\tau$ $\widetilde{A_{k\pm}}\big\rvert_{\xi} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{\tau\to-\infty}^{\tau\to\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{i\omega\frac{\xi}{c}\eta}\,d\eta$ We define $\boxed{\nu \overset{\text{def}}{=} \lvert\xi\rvert\frac{\omega}{c}}$ ... (138) Because of $\eta = \frac{c}{\xi}\tau$ the integration boundaries w.r.t. η are different depending on the sign of ξ: $\widetilde{A_{k\pm}}\big\rvert_{\xi>0} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{i\xi\frac{\omega}{c}\eta}\,d\eta \xRightarrow{(138a)} \boxed{\widetilde{A_{k\pm}}\big\rvert_{\xi>0} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{i\nu\eta}\,d\eta}$ ... $(139a)$ $\widetilde{A_{k\pm}}\big\rvert_{\xi<0} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{\infty}^{-\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{i\xi\frac{\omega}{c}\eta}\,d\eta \xRightarrow{(138a)} \boxed{\widetilde{A_{k\pm}}\big\rvert_{\xi<0} = -\sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(\pm ik\xi e^{\mp\eta})\,e^{-i\nu\eta}\,d\eta}$ ... $(139b)$
Fourier Transformed Modes Right observer, right moving waves (+ index)	$\widetilde{A_{k+}}\big\rvert_{R} = \widetilde{A_{k+}}\big\rvert_{\xi>0} \overset{(139a)}{=} \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(ik\xi e^{-\eta})\,e^{i\nu\eta}\,d\eta = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(ik\xi e^{-\eta})\,e^{(-\eta)(-i\nu)}\,d\eta$ $\underline{\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(ik\xi e^{-\eta})\,(e^{-\eta})^{-i\nu}\,d\eta}$ ... (140) $\underline{y \overset{\text{def}}{=} -ik\xi e^{-\eta}}$ ... $(141a) \Longrightarrow \underline{e^{-\eta} = -\frac{y}{ik\xi} = i(k\xi)^{-1}y}$ ... $(141b)$ $(141a) \Longrightarrow \frac{dy}{d\eta} = ik\xi e^{-\eta} \overset{(141a)}{=} -y \Longrightarrow \underline{d\eta = -y^{-1}\,dy}$ ... $(141c)$ $\underline{y(-\infty) \overset{(141a)}{=} -i\infty,\ \ y(+\infty) \overset{(141a)}{=} 0}$ ... $(141d)$ $(141abcd) \xRightarrow{(140)} \widetilde{A_{k+}}\big\rvert_{R} = -\sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-i\infty}^{0} e^{-y}(i(k\xi)^{-1}y)^{-i\nu}y^{-1}\,dy = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{0}^{-i\infty} e^{-y}(i(k\xi)^{-1}y)^{-i\nu}y^{-1}\,dy \Longrightarrow$ $\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}(i(k\xi)^{-1})^{-i\nu}\int_{0}^{-i\infty} e^{-y}y^{-i\nu}y^{-1}\,dy = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}i^{-i\nu}(k\xi)^{i\nu}\int_{0}^{-i\infty} e^{-y}y^{-i\nu}y^{-1}\,dy \Big\vert\ i^{-i\nu} = ((e^{i\pi/2})^{-i})^{\nu} = e^{\nu\pi/2}$ $\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}\int_{0}^{-i\infty} e^{-y}y^{-i\nu}y^{-1}\,dy \Big\vert\ s = iy \Longrightarrow y = \frac{s}{i} = -is \Longrightarrow dy = -i\,ds \Longrightarrow s(-i\infty) = \infty$ $\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}\int_{0}^{\infty} e^{is}(-is)^{-i\nu}(-is)^{-1}(-i)\,ds = \cancel{\frac{-i}{-i}}\sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}(-i)^{-i\nu}\int_{0}^{\infty} e^{is}s^{-i\nu}s^{-1}\,ds$ $\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}((e^{-i\pi/2})^{-i})^{\nu}\int_{0}^{\infty} e^{is}s^{-i\nu}s^{-1}\,ds = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}\underbrace{e^{-\nu\pi/2}\int_{0}^{\infty} e^{is}s^{-i\nu}s^{-1}\,ds}_{\Gamma(-i\nu)}$ $\boxed{\widetilde{A_{k+}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{i\nu}\,\Gamma(-i\nu)}$ ... (142)
Fourier Transformed Modes Right observer, left moving waves (− index)	$\widetilde{A_{k-}}\big\rvert_{R} = \widetilde{A_{k-}}\big\rvert_{\xi>0} \overset{(139a)}{=} \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(-ik\xi e^{\eta})\,e^{i\nu\eta}\,d\eta$ $\underline{\widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{-\infty}^{\infty}\exp(-ik\xi e^{\eta})\,(e^{\eta})^{i\nu}\,d\eta}$ ... (143) $\underline{y \overset{\text{def}}{=} ik\xi e^{\eta}}$ ... $(144a) \Longrightarrow \underline{e^{\eta} = \frac{y}{ik\xi} = -i(k\xi)^{-1}y}$ ... $(144b)$ $(144a) \Longrightarrow \frac{dy}{d\eta} = ik\xi e^{\eta} \overset{(144a)}{=} y \Longrightarrow \underline{d\eta = y^{-1}\,dy}$ ... $(144c)$ $\underline{y(-\infty) \overset{(144a)}{=} 0,\ \ y(+\infty) \overset{(144a)}{=} i\infty}$ ... $(144d)$ $(144abcd) \xRightarrow{(143)} \widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}\int_{0}^{i\infty} e^{-y}(-i(k\xi)^{-1}y)^{i\nu}\,y^{-1}\,dy = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}(-i)^{i\nu}(k\xi)^{-i\nu}\int_{0}^{i\infty} e^{-y}y^{i\nu}\,y^{-1}\,dy \Longrightarrow$ $\widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}((e^{-i\pi/2})^{i})^{\nu}(k\xi)^{-i\nu}\int_{0}^{i\infty} e^{-y}y^{i\nu}\,y^{-1}\,dy = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}\int_{0}^{i\infty} e^{-y}y^{i\nu}\,y^{-1}\,dy$ $s = -iy \Longrightarrow y = -\frac{s}{i} = is \Longrightarrow dy = i\,ds \Longrightarrow s(i\infty) = \infty \Longrightarrow \widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}\int_{0}^{\infty} e^{-is}(is)^{i\nu}(is)^{-1}i\,ds$ $\widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}i^{i\nu}\int_{0}^{\infty} e^{-is}s^{i\nu}s^{-1}\,ds = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}((e^{i\pi/2})^{i})^{\nu}\int_{0}^{\infty} e^{-is}s^{i\nu}s^{-1}\,ds$ $\widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}\underbrace{e^{-\nu\pi/2}\int_{0}^{\infty} e^{-is}s^{i\nu}s^{-1}\,ds}_{\Gamma(i\nu)} \Longrightarrow \boxed{\widetilde{A_{k-}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{-i\nu}\,\Gamma(i\nu)}$ ... (145)
Right observer, both waves	$(142)(145) \Longrightarrow \boxed{\widetilde{A_{k\pm}}\big\rvert_{R} = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}}\frac{\xi}{c}e^{\nu\pi/2}(k\xi)^{\pm i\nu}\,\Gamma(\mp i\nu)}$ ... (146) Note: Strictly speaking, the Gamma-function of a purely imaginary number is divergent. This is a case of “renormalization in disguise” (also known to some as “ghetto-math”).

Right observer, Complex conjugate	To do the mode expansion (24), we also need expressions for the complex conjugates $\left.\widetilde{A^*_{k\pm}}\right\vert_R$. The calculation is analogous to before, and leads to the result $\boxed{\left.\widetilde{A^*_{k\pm}}\right\vert_R = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}\frac{\xi}{c}}\,e^{-\nu\pi/2}(k\xi)^{\pm i\nu}\,\Gamma(\mp i\nu)}$ ... (147) $\overset{(146)}{\Longrightarrow}$ $\boxed{\left.\widetilde{A^*_{k\pm}}\right\vert_R = e^{-\nu\pi}\left.\widetilde{A_{k\pm}}\right\vert_R}$ ... (148) This means that for the right Rindler observer negative-frequency modes are strongly related to positive-frequency modes.
Left observer, all Rindler observers	A calculation analogous to the calculations before yields the following results for the left observer: $\left.\begin{array}{l}\boxed{\left.\widetilde{A_{k\pm}}\right\vert_L = \left.\widetilde{A_{k\pm}}\right\vert_{\xi<0} = \cdots = -\sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}\frac{\xi}{c}}\,e^{\nu\pi/2}(k\xi)^{\mp i\nu}\,\Gamma(\pm i\nu)}\ \ldots(149a)\\ \boxed{\left.\widetilde{A^*_{k\pm}}\right\vert_L = \left.\widetilde{A^*_{k\pm}}\right\vert_{\xi<0} = \cdots = -\sqrt{\frac{\hbar}{4\pi\varepsilon_0 ck}\frac{\xi}{c}}\,e^{-\nu\pi/2}(k\xi)^{\mp i\nu}\,\Gamma(\pm i\nu)}\ \ldots(149b)\end{array}\right\}\Longrightarrow \boxed{\left.\widetilde{A^*_{k\pm}}\right\vert_L = e^{-\nu\pi}\left.\widetilde{A_{k\pm}}\right\vert_L}$ ... (150) Also for the left Rindler observer negative-frequency modes are strongly related to positive-frequency modes. Therefore we can write for all Rindler observers :(148)(150) $\Longrightarrow$ $\boxed{\left.\widetilde{A^*_{k\pm}}\right\vert_\xi = e^{-\nu\pi}\left.\widetilde{A_{k\pm}}\right\vert_\xi}$ ... (151)
Vacuum waves connect both observers	Consider, w.l.o.g., a right-moving wave packet $A(q) = \int_0^\infty \tilde{A}(k)\,e^{ikq}\,dk$ with $q \overset{\text{def}}{=} x - ct$. In order for the wave-packet $A(q)$ to show physical behavior for $q \in \mathbb{R}$, the Fourier transform $\tilde{A}(k)$ has to be such that $A(q)$ is analytic for $\mathrm{Im}(q) \geq 0$. Analytic functions cannot vanish on a finite interval of the real line, or otherwise the function has to vanish everywhere. This implies that the positive-frequency wave packets are necessarily infinitely extended and are therefore bridging the gap between both Rindler wedges.
Annihilation operators for observers at rest and Rindler observers	Note that for the left observer, the proper time τ runs against η. Therefore we associate negative frequencies $\nu < 0$ with the left part of the space-time diagram. Based on this, we define the following photon annihilation operators: $\hat{a}_\nu$ ... photon annihilation operator for observer at rest on the right side of the space-time diagram $\hat{a}_{-\nu}$ ... photon annihilation operator for observer at rest on the left side of the space-time diagram The Minkowsky vacuum state $\vert 0\rangle_{\mathrm{M}}$ for observers at rest is defined by $\hat{a}_{\pm\nu}\vert 0\rangle_{\mathrm{M}} = 0$. The annihilation operators for the left and right Rindler observers ($\widehat{b}_\nu$ and $\widehat{b}_{-\nu}$) are found by a Bogoliubov transformation: $\boxed{\widehat{b}_\nu = \cosh(\zeta)\,\hat{a}_\nu + \sinh(\zeta)\,\hat{a}^\dagger_{-\nu}}$ ... (152a) $\boxed{\widehat{b}_{-\nu} = \cosh(\zeta)\,\hat{a}_{-\nu} + \sinh(\zeta)\,\hat{a}^\dagger_\nu}$ ... (152b) with $\boxed{\tanh(\zeta) = e^{-\pi\nu}}$ ... (153c) Inverse transformation: $\boxed{\hat{a}_{\pm\nu} = \cosh(\zeta)\,\widehat{b}_{\pm\nu} - \sinh(\zeta)\,\widehat{b}^\dagger_{\mp\nu}}$ ... (153d)
$\vert 0\rangle_{\mathrm{M}}$ appears as EPR state to Rindler observers	We write $\vert 0\rangle_{\mathrm{M}}$ in the product Fock basis of the two Rindler observers and apply $\hat{a}_\nu$: $\vert 0\rangle_{\mathrm{M}} = \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Big\vert \hat{a}_\nu \cdot \Rightarrow \hat{a}_\nu\vert 0\rangle_{\mathrm{M}} = \hat{a}_\nu \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Rightarrow 0 = \hat{a}_\nu \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \overset{(153d)}{\Longrightarrow}$ $0 = \left(\cosh(\zeta)\,\widehat{b}_\nu - \sinh(\zeta)\,\widehat{b}^\dagger_{-\nu}\right)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\widehat{b}_\nu\vert n_L,n_R\rangle - \sinh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\widehat{b}^\dagger_{-\nu}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=1}^\infty c_{n_L,n_R}\sqrt{n_R}\vert n_L,n_R-1\rangle - \sinh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\sqrt{n_L+1}\vert n_L+1,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=1}^\infty c_{n_L,n_R}\sqrt{n_R}\vert n_L,n_R-1\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=0}^\infty c_{n_L-1,n_R}\sqrt{n_L}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=1}^\infty c_{n_L,n_R}\sqrt{n_R}\vert n_L,n_R-1\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L-1,n_R-1}\sqrt{n_L}\vert n_L,n_R-1\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_R=0}^\infty c_{0_L,n_R}\sqrt{n_R}\vert 0,n_R-1\rangle +$ $\quad\cosh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L,n_R}\sqrt{n_R}\vert n_L,n_R-1\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L-1,n_R-1}\sqrt{n_L}\vert n_L,n_R-1\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_R=0}^\infty c_{0_L,n_R}\sqrt{n_R}\vert 0,n_R-1\rangle + \sum_{n_L=1}^\infty \sum_{n_R=1}^\infty\left(\cosh(\zeta)\,c_{n_L,n_R}\sqrt{n_R} - \sinh(\zeta)\,c_{n_L-1,n_R-1}\sqrt{n_L}\right)\vert n_L,n_R-1\rangle$ ... (154) In order for this linear combination to vanish, all coefficients have to be zero: $\forall n_L,n_R \geq 1{:}\ \cosh(\zeta)\,c_{n_L,n_R}\sqrt{n_R} - \sinh(\zeta)\,\sqrt{n_L}c_{n_L-1,n_R-1} = 0 \,\Big\vert\cdot\frac{1}{\sqrt{n_L}\cosh(\zeta)} \Longrightarrow$ $c_{n_L,n_R}\sqrt{\frac{n_R}{n_L}} - \tanh(\zeta)\,c_{n_L-1,n_R-1} = 0 \Longrightarrow \boxed{c_{n_L,n_R}\sqrt{\frac{n_R}{n_L}} = \tanh(\zeta)\,c_{n_L-1,n_R-1}}$ ... (155) Again, we write $\vert 0\rangle_{\mathrm{M}}$ in the product Fock basis of the two Rindler observers and now apply $\hat{a}_{-\nu}$: $\vert 0\rangle_{\mathrm{M}} = \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Big\vert \hat{a}_{-\nu} \cdot\!\Longrightarrow \hat{a}_{-\nu}\vert 0\rangle_{\mathrm{M}} = \hat{a}_{-\nu} \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \hat{a}_{-\nu}\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \overset{(153d)}{\Longrightarrow} 0 = \left(\cosh(\zeta)\,\widehat{b}_{-\nu} - \sinh(\zeta)\,\widehat{b}^\dagger_{+\nu}\right)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\widehat{b}_{-\nu}\vert n_L,n_R\rangle - \sinh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\widehat{b}^\dagger_{+\nu}\vert n_L,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\sqrt{n_L}\vert n_L-1,n_R\rangle - \sinh(\zeta)\sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\sqrt{n_R+1}\vert n_L,n_R+1\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\sqrt{n_L}\vert n_L-1,n_R\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=0}^\infty c_{n_L-1,n_R}\sqrt{n_R+1}\vert n_L-1,n_R+1\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\sqrt{n_L}\vert n_L-1,n_R\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L-1,n_R-1}\sqrt{n_R}\vert n_L-1,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=1}^\infty c_{n_L,0_R}\sqrt{n_L}\vert n_L-1,0\rangle +$ $\quad\cosh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L,n_R}\sqrt{n_L}\vert n_L-1,n_R\rangle - \sinh(\zeta)\sum_{n_L=1}^\infty \sum_{n_R=1}^\infty c_{n_L-1,n_R-1}\sqrt{n_R}\vert n_L-1,n_R\rangle \Longrightarrow$ $0 = \cosh(\zeta)\sum_{n_L=1}^\infty c_{n_L,0_R}\sqrt{n_L}\vert n_L-1,0\rangle + \sum_{n_L=1}^\infty \sum_{n_R=1}^\infty\left(\cosh(\zeta)\,c_{n_L,n_R}\sqrt{n_L} - \sinh(\zeta)\,c_{n_L-1,n_R-1}\sqrt{n_R}\right)\vert n_L-1,n_R\rangle$ ... (156) In order for this linear combination to vanish, all coefficients have to be zero: $\forall n_L,n_R \geq 1{:}\ \cosh(\zeta)\,c_{n_L,n_R}\sqrt{n_L} - \sinh(\zeta)\,c_{n_L-1,n_R-1}\sqrt{n_R} = 0 \,\Big\vert\cdot\frac{1}{\sqrt{n_r}\cosh(\zeta)} \Longrightarrow$ $c_{n_L,n_R}\sqrt{\frac{n_L}{n_R}} - \tanh(\zeta)\,c_{n_L-1,n_R-1} = 0 \Longrightarrow \boxed{c_{n_L,n_R}\sqrt{\frac{n_L}{n_R}} = \tanh(\zeta)\,c_{n_L-1,n_R-1}}$ ... (157) (157) – (155) $\Longrightarrow \forall n_L,n_R \geq 1{:}\ c_{n_L,n_R}\left(\sqrt{\frac{n_L}{n_R}} - \sqrt{\frac{n_R}{n_L}}\right) = 0 \Longrightarrow \frac{n_L}{n_R} = \frac{n_R}{n_L} \Longrightarrow \boxed{\forall n_L \neq n_R \Longrightarrow c_{n_L,n_R} = 0}$ ... (158a) $\Longrightarrow \forall n_L = n_R \overset{(155)}{\Longrightarrow} c_{n,n} = \tanh(\zeta)\,c_{n-1,n-1} \Longrightarrow \boxed{c_{n,n} = c_{0,0}\tanh^n(\zeta)}$ ... (158b) ${}_M\langle 0\vert 0\rangle_M = 1 \Longrightarrow 1 = \sum_{n=0}^\infty c^2_{n,n} \overset{(158b)}{\Longrightarrow}$ $1 = c^2_{0,0}\sum_{n=0}^\infty \tanh^{2n}(\zeta) = c^2_{0,0}\sum_{n=0}^\infty(\tanh^2(\zeta))^n = \frac{c^2_{0,0}}{1-\tanh^2(\zeta)} \Longrightarrow 1 = c^2_{0,0}\cosh^2(\zeta) \Longrightarrow \boxed{c_{0,0} = \frac{1}{\cosh(\zeta)}}$ ... (158c) $\vert 0\rangle_{\mathrm{M}} = \sum_{n_L=0}^\infty \sum_{n_R=0}^\infty c_{n_L,n_R}\vert n_L,n_R\rangle \overset{(158a)}{=} \sum_{n=0}^\infty c_{nn}\vert n_L,n_R\rangle \overset{(158b)}{=} c_{0,0}\sum_{n=0}^\infty \tanh^n(\zeta)\,\vert n_L,n_R\rangle \overset{(158c)}{\Longrightarrow}$ $\vert 0\rangle_{\mathrm{M}} = \frac{1}{\cosh(\zeta)}\sum_{n=0}^\infty \tanh^n(\zeta)\,\vert n_L,n_R\rangle \overset{(153c)}{\Longrightarrow} \boxed{\vert 0\rangle_{\mathrm{M}} = \frac{1}{\cosh(\zeta)}\sum_{n=0}^\infty e^{-n\pi\nu}\vert n_L,n_R\rangle}$ ... (159) ... appears as EPR state!

Interpretation	The Minkowsky vacuum $\|0\rangle_{\mathrm{M}}$ appears to the accelerated Rindler observers as entangled EPR state with strong correlations between the observers L and R. These correlations are not linked to a specific point in time, but materialize in the recorded spectra by integration over time. The explanation for this is that the vacuum fluctuations cannot be confined to a subpart of spacetime. Both Rindler observers see the fluctuations from the same wave.
Unruh-Fulling-Davies temperature	From the perspective of a Rindler observer, the Minkowsky vacuum $\|0\rangle_{\mathrm{M}}$ appears as thermal state. $\boxed{\hat{\rho}_R = \mathrm{tr}_L\big(\|0\rangle_{\mathrm{MM}}\langle 0\|\big) \overset{(159)}{=} \frac{1}{\cosh^2(\zeta)} \sum_{n=0}^{\infty} e^{-2n\pi\nu} \|n_R\rangle\langle n_L\|}$ ... (160) To assign a temperature to this state, we identify: $2\pi\nu = \frac{\hbar\omega}{k_B T} \overset{(138)}{\Longrightarrow} 2\pi \frac{\|\xi\|\omega}{c} \frac{c}{c} = \frac{\hbar\omega}{k_B T} \Longrightarrow 2\pi c \frac{\|\xi\|}{c^2} = \frac{\hbar}{k_B T} \overset{(122)}{\Longrightarrow} 2\pi c \frac{1}{a} = \frac{\hbar}{k_B T} \Longrightarrow \boxed{k_B T = \frac{\hbar a}{2\pi c}}$ ... (161) ✓cf. (135)